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P. A. Zhilin

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Selection of articles presented at
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In this selection you may find presentations by P.A. Zhilin given from 1994 to 2005 at the International Summer School – Conference "Advanced Problems in Mechanics". This book consists of two volumes: the first one containes articles in Russian, and the second one in English. The scope of the questions under discussion is wide. It includes fundamental laws of mechanics, direct tensor calculus, dynamics of rigid bodies, nonlinear theory of rods, general theory of inelastic media including the theory of plasticity, consolidating granular media, phase transitions, and also piezoelectricity, ferromagnetism, electrodynamics, and quantum mechanics. This selection of papers by P.A. Zhilin, in fact, presents method of construction of continual theories with rotational degrees of freedom, mathematical technique necessary for this purpose, and also examples of application of the theories mentioned above to the description of various physical phenomena.

For researchers, PhD students, undergraduate students of last years specializing in mechanics and theoretical physics.

ISBN

Introduction

In this selection you can find the papers, presented either by P.A. Zhilin or by his co-authors at the APM Summer-School Conference in the period from 1994 to 2005. The book is issued in two volumes: the first one contains articles in Russian, the second one contains articles in English. In both volumes the articles are listed in chronological order. The range of questions discussed is wide. It includes fundamental laws of mechanics, direct tensor calculus, rigid body dynamics, nonlinear rod theory, general theory of non-linear media, including plasticity, consolidating granular media, phase transitions, as well as piezoelectricity, ferromagnetism, electrodynamics and quantum mechanics. At first sight it seems that the papers are not related one to another. But this is not so. Let us show a few examples. Rigid body oscillator, introduced in the article related to the absolute rigid body dynamics, is used further as fundamental model when constructing inelastic media theory, piezoelectricity theory, and theory of magnetoelastic materials. Methods of description of the spinor motion, based on use of the direct tensor calculus, are used and developed both for solving rigid body dynamics problems and for solving nonlinear rod theory problems. The same methods are used when constructing various continuum models, which take into account rotational degrees of freedom. The symmetry theory and tensor invariant theory, which are presented in the book, dedicated to this topic, are being actively used and developed when constructing rod theory, as well as for other continuum theories. Two papers are dedicated to the formulation of fundamental laws of the Eulerian mechanics — mechanics of a general body, consisting of particles with rotational degrees of freedom. All continuum theories, presented in the digest, including electrodynamics, are built adhering to the same positions based on the fundamental laws of mechanics. When building continuum models both for elastic and inelastic media, the theory of strains is used, which is based on the idea of using the reduced energy balance equation for defining measures of deformation. By the elementary examples of discrete systems mechanics the notions of internal energy, chemical potential, temperature and entropy are introduced. Definition of these quantities is given by means of pure mechanical arguments, which are based on using special mathematical formulation of energy balance equation. The same method of introducing the basic thermodynamics notions indicated above is used when building different continuum theories. In fact the selected papers of P.A. Zhilin represent the method for constructing continuum theories with rotational degrees of freedom together with the necessary mathematical apparatus, as well as examples of using the mentioned theories when describing different physical phenomena. Among others, the first volume of the digest includes two papers, dedicated to the fundamental laws of mechanics, which were written with big time interval, and two articles on the rod theory, which were also written in the different periods of time. The Reader can take advantage of following the development of scientific ideas. The first paper dedicated to the fundamental laws of mechanics, is a quite perfect, logically rigorous theory. Nevertheless, after many years, author returns to this topic. The aim was not to change something in the original variant, but to complete it by including in it thermodynamical ideas. The mentioned above can equally be pertained to the two papers on the rod theory. Not every physical theory permits including of new notions in it. Often, when needed to describe a new phenomenon, one is forced to reject an old theory and build a new one instead. The theories presented in this selection have an ability to be developed. This is their great advantage, and that is one of the important reasons why they attract attention of researchers.

The editorial board is grateful to N.A. Zhilina for help with preparing this book for publication; to S.N. Gavrilov, E.F. Grekova, I.I. Neygebauer, and E.V. Shishkina for translation to English of the introductional parts and the list of publications.

D.A. Indeitsev, E.A. Ivanova, A.M. Krivtsov.

Contents

P. A. Zhilin — searching for Truth

*"There is no action without reason in nature;
comprehend the reason and you won't need the experience."*

Leonardo da Vinci

The most significant features of scientific society in the end of XX — beginning of XXI century are pragmatism and particular specialization. To the least degree this can be applied to Pavel Andreevich Zhilin. Sincere interest, willing to perceive the Truth and to bring his knowledge to people were the solely motives for his work. Breadth of scientific interests of P.A. Zhilin is impressing — having fundamental character, his works cover practically all areas of mechanics and are extended to electrodynamics and quantum physics. Hardly anyone can express the views of P.A. Zhilin on science better than himself:

"Aim of each science is in perception of the Reality. At the same time the science investigates not the Reality itself, but the simplified models of the Reality. Approaching to the true Reality can be achieved by broadening the model. But to construct a model we need to know at least, what exactly we are going to model. In other words we have to have an a priori idea of the Reality. So we have a vicious circle — to perceive the Reality we need a science, and to construct a science we need to know the Reality. Fortunately the solution of this one would think unsolvable problem is integrated in the human mind, which has two qualitatively different categories: a) intuition and b) intellect.

Intuition is the ability of a human being to sense the world around us directly, which can not be reduced to the five basic senses. This is what poets, musicians, painters and other artists are conscious of. Intuition may be trained as well as every other ability of human being, but it requires permanent and purposeful efforts.

Intellect is an ability of human being to think logically, basing on an a priori knowledge, "built in" the intellect "memory". A powerful modern computer is a practically perfect analogue of intellect."

From the paper "Reality and Mechanics"

Doctor of Science, professor, author of more than 200 scientific papers, many of which were published in the key scientific journals, a Teacher who educated more than one generation of disciples — both PhD and Dr. Sci, P.A. Zhilin was a mind of a wide scope and of great erudition.

Being by his position an adherent of the rigorous Science, he was also deeply interested in Eastern philosophies. Fundamental scientific ideas of Pavel Andreevich, concerning the importance of spinor motions when describing events at the micro-level and modelling the electromagnetic field, are in correspondence with different metaphysical concepts of the origin of the World. These ideas in various forms were proposed by the great classics of science, whose works Pavel Andreevich studied in a deep and detailed way. The achievement of Pavel Andreevich is the translation of these ideas from a vague general form of words and intuitive assumptions into a rigourous form of mathematical models. The things he writes on intuitive perception of the world around us is based not only on books, but on his own experience of direct perception of scientific knowledge:

"It is principally possible to use intuitive and intellectual methods of perception independently one of another. Intuitive perception has an imperfection of being impossible to teach it. However namely the intuitive method underlies the creation of scientific models. Pure intellectual approach can make semblance of scientific discoveries, but in fact it's fruitless. In the last decades special popularity was gained by the so called "black box" philosophy, which refers to the intellectual method achievement. It seemed that this way could bring us to success. But in actual fact it turned out that the black box is worth only when it is transparent, that is when we know its inside beforehand. The merit of the intellectual method is that it can be taught easily.

Let us characterize the intellectual method with the words of Einstein: "Science is a creation of human mind with its freely invented ideas and notions".

Intuitive method of cognition is best defined by the words of Socratus: By intuitive perception "soul is climbing up the highest observation tower of Being".

The main thesis of this work is that no real development of science is possible without immediate participation of intuition and there are neither freely invented ideas nor notions existing in nature.."

From the paper "Reality and Mechanics"

Having administrative positions of the head of the Chair of Theoretical Mechanics at the Saint Petersburg Polytechnical University, head of laboratory "Mechanical systems dynamics" at the Institute for Problems in Mechanical Engineering Russian Academy of Sciences, taking active part in the life of society — being member of the Russian National Committee for Theoretical and Applied Mechanics, member of International Society of Applied Mathematics and Mechanics (GAMM), member of Guidance Board Presidium for Applied Mechanics Ministry of Higher Education RF, full member of Russian Academy of Sciences for durability problems, member of three Dissertation Councils, first of all P.A. Zhilin was a Scientist, for whom the science has become the sense of life and the cause of life. He was a Teacher who influenced not only his immediate disciples — PhD students and persons working for doctor's degree, but also many people considering themselves his disciples to a greater or lesser extent.

P.A. Zhilin considered one of his main tasks broadening the range of application for mechanics and describing phenomena, being studied in the different fields of natural science from common rational positions, peculiar to mechanics. The following quotation expresses the views of P.A. Zhilin on mechanics as a method of studying nature and on the role mechanics should play in the science of XXI century:

"Mechanics is not a theory of whatever Phenomenon, but a method of investigation of nature. There is no law in the foundations of mechanics, which could be disproved experimentally, not even in principle. In the foundation of mechanics there are logical statements which express balance conditions for certain quantities, and per se they are insufficient for the construction of any closed theory. One has to attract supplementary laws, like the law of gravity, regarded as facts experimentally determined. These supplementary laws may come out to be insufficient or even erroneous, but rejecting them does not influence methods of mechanics. The mentioned nonclosure of mechanics may be considered as its loss by people who think that the humanity is close to the final understanding of the universe. But those who are able to see the Reality, understand how infinitely far people are from ability to describe even relatively simple phenomena of the Reality. That is why the correct method of studying nature is to include a priori indefinite elements, manipulating by which one could improve these or those theories of phenomena of various nature and in that way broaden our idea of Reality. Mechanics sets certain limits for the acceptable structure of these indefinite elements, but preserves a wide enough freedom for them."

From the paper "Reality and Mechanics"

One of the most important results of the scientific and educational work of P.A. Zhilin is his book of about 1000 pages, which was published only partly during his life. The book represents a course of the Eulerian mechanics, which takes into account on equal terms both translational and rotational degrees of freedom. In this book P.A. Zhilin shares with the reader his ideas related to the taking into consideration spinor motions on the micro-level, application of open bodies models, and introduction of the characteristics of physical state (temperature, entropy, chemical potential) by methods of rational mechanics.

P.A. Zhilin dreamed to open a way to the microworld for the rational mechanics, and to include there the electrodynamics. Many people dream and many people issue big challenges for themselves, but only few of them succeed. P.A. Zhilin knew to make his dreams a reality. Within the limits of classical mechanics he offered continuum models, whose mathematical description is coming to electrodynamics and quantum mechanics equations. Views of P.A. Zhilin often disagree with the common point of view, his ideas are raising debates, but

"Who argues, appealing to an authority, uses not his brain, but rather his memory."

Leonardo da Vinci

D.A. Indeitsev, E.A. Ivanova, A.M. Krivtsov

Short biography and scientific results of P. A. Zhilin*

Pavel Andreevich Zhilin was the Head of the Department of Theoretical Mechanics at Saint Petersburg Polytechnical University, Head of the laboratory “Dynamics of Mechanical Systems” at the Institute for Problems in Mechanical Engineering of Russian Academy of Sciences, member of the Russian National Committee for Theoretical and Applied Mechanics, member of the International Society of Applied Mathematics and Mechanics (GAMM), member of Guidance Board Presidium for Applied Mechanics Ministry of Higher Education RF, full member of Russian Academy of Sciences for Strength Problems. He was an author of more than 200 scientific papers, monographs “Second-rank Vectors and Tensors in 3-dimentional space” (2001), “Theoretical mechanics: fundamental laws of mechanics” (2003). Sixteen PhD theses and six Professorial theses were defended under his supervision.

P.A. Zhilin was born on February 8th, 1942, in the town of Velikiy Ustyug in Vologda region, where his family found themselves during the war. Pavel Zhilin spent his childhood in the towns of Volkhov and Podporozhie, where his father, Andrey Pavlovich Zhilin, worked. Andrey P. Zhilin was a power engineering specialist, and at that time the chief engineer at the coordinated hydroelectric system of Svir river. Zoya Alexeevna Zhilina, mother of Pavel A. Zhilin, was bringing up the sons and kept the house. In 1956 Andrey P. Zhilin was assigned to the position of the chief power engineering specialist at the Soviet Union Trust “HydroElectroMontage”, and the family moved to Leningrad. The elder brother, Sergey Andreevich Zhilin, followed in his father’s footsteps, became an engineer and now participates in creating high-voltage electric apparatus. In 1959 P.A. Zhilin left the secondary school and entered Leningrad Polytechnical Institute. Yet at school Pavel Zhilin met his future wife, Nina Alexandrovna, who was his faithful friend and helpmate all his life long. While studying at the institute P.A. Zhilin became keen on table tennis and was a captain of the student and later institute team for many years. Not once did the team win different student and sport collectives championships. P.A. Zhilin got a qualification of the candidate master of sports (the highest qualification in this sport discipline at that time).

In the period of 1959–1965 P.A. Zhilin studied at Leningrad Polytechnical Institute

*The editorial board is grateful to N.A. Zhilina for the biografic data of P.A. Zhilin. In the survey of scientific results we used, when it was possible, the original text of manuscipts and articles by P.A. Zhilin.

in the Department of "Mechanics and Control Processes" at the Faculty of Physics and Mechanics. Later on his daughter, Olga Zhilina, graduated from the same Department. After graduation, P.A. Zhilin got a qualification of engineer-physicist in "Dynamics and Strength of Machines" speciality, and from 1965 to 1967 worked as an engineer at water turbine strength department in the Central Boiler Turbine Institute. In 1967 he accepted a position of Assistant Professor at the Department of "Mechanics and Control Processes", later he worked there as a senior researcher, an Associate Professor and a Full Professor. The founder of the Chair was Anatoliy Isaakovich Lurie, Doctor of Technical Sciences, Professor, corresponding member of USSR Academy of Sciences, world-famous scientist. P.A. Zhilin became the closest disciple of A.I. Lurie and spent many hours working together with him. Scientific ideology of P.A. Zhilin was developing to a great extent under the influence of A.I. Lurie. P.A. Zhilin got his PhD degree in Physical and Mathematical Sciences in 1968 (the topic of his thesis was "The theory of ribbed shells"), Professor of Physical and Mathematical Sciences since 1984 (the topic of his Professorial thesis was "The theory of simple shells and its applications"), Professor at the Department of "Mechanics and Control Processes" since 1989. In 1974–1975 P.A. Zhilin worked as a visiting researcher at the Technical University of Denmark. While working in the Department of "Mechanics and Control Processes", P.A. Zhilin delivered lectures on analytical mechanics, theory of oscillations, theory of shells, tensor analysis, continuum mechanics. In 1988 he was invited in the Yarmuk University (Jordan) to set a course of continuum mechanics at the Faculty of Physics. At the same time P.A. Zhilin actively carried out scientific work in the field of theory of plates and shells, nonlinear rod theory, theory of elasticity, continuum mechanics. He gained three certificates of invention in the area of vibroinsulation and hydroacoustics, he was awarded with the Inventor of the USSR insignia.

Since 1989 P.A. Zhilin was the Head of Department of Theoretical Mechanics. In the period of his direction five of his colleagues defended their Professorial theses, for the four of them P.A. Zhilin was a scientific advisor. While working in the Department of Theoretical Mechanics P.A. Zhilin stationed and read original courses on tensor algebra, rational mechanics, and the rod theory. During this period of time Pavel Zhilin worked hard in the field of investigating and developing foundations of mechanics. His investigations on spinor motions in mechanics and physics, phase transitions and phenomena of inelasticity, electrodynamics from the positions of rational mechanics, logical foundations of mechanics relate to this period. Since 1994 Pavel Zhilin was the Head of "Dynamics of Mechanical Systems" laboratory at the Institute for Problems in Mechanical Engineering of Russian Academy of Sciences. Since 1999 he was a member of the scientific committee of the Annual International Summer School – Conference "Advanced Problems in Mechanics", held by the Institute for Problems in Mechanical Engineering.

Pavel Andreevich Zhilin died on 4th of December, 2005. His track has become a part of history of science. It is difficult to overestimate his influence on his disciples, colleagues, and all who were lucky to know him personally. He had an extraordinary ability to inspire interest to science, to give you a fresh unexpected look at the world around. P.A. Zhilin was a man of heart, a responsive, kind person, who found time for everyone, always giving his full support and benefit of his wise advice. One was amazed by his remarkable human qualities, his absolute scientific and human honesty. Being his disciples we are grateful to life for the chance to have known such a wonderful person and an outstanding scientist, who became for us an embodiment of spirituality.

Scientific results

Theory of shells

Zhilin's early works, Ph.D. and Professor theses were devoted to the development of the theory of shells. When Zhilin started his research in this area, there existed no general theory of shells. For each class of shell-type constructions there were developed particular independent theories: the theory of thin single-layer shells; the theory of engineering anisotropic shells; the theory of ribbed shells; the theory of thin multi-layer shells; the theory of perforated shells; the theory of cellular shells; the theory of thick single-layer shells, and many others. Within each theory one could distinguish several versions, which differed in basic assumptions as well as in final equations. The theories of shells are still being developed since the science gives birth to new constructions that can not be described within the existing variety of theories. Zhilin created (1975–1984) the general nonlinear theory of thermoelastic shells, whose way of construction fundamentally differs from the one of all known versions of shell theories, and can be easily generalised for any shell-like constructions and other objects of continuum mechanics. This approach is comprehensively described in work [1].

1. Zhilin P.A. Applied mechanics. Foundations of theory of shells. Tutorial book. St. Petersburg State Polytechnical University. 2006. 167 p. (In Russian).

Discretely stiffened thermoelastic shells

The general theory of discretely stiffened thermoelastic shells was developed (1965–1970) [1, 3] and applied to the following practical problems: the calculation of the high-pressure water turbine scroll of Nurek hydropower station [2] and of the vacuum chamber of the thermonuclear Tokamak 20 Panel [4].

There was proposed (1966) a variant of the Steklov-Fubini method for differential equations, whose coefficients have singularities of δ-function type. The method allowed to find the solution in an explicit form for the problem of axisymmetric deformation of a discretely stiffened cylindrical shell [5].

1. Zhilin P.A. General theory of ribbed shells // Trudi CKTI (Transactions of Central Boiler Turbine Institute). 1968. No. 88. Pp. 46–70. (In Russian.)

2. Zhilin P.A., Mikheev V.I. Toroidal shell with meridional ribs for design of hydroturbine spirals. // Trudi CKTI (Transactions of Central Boiler Turbine Institute). 1968. No. 88. Pp. 91–99. (In Russian.)

3. Zhilin P.A. Linear theory of ribbed shells // Izvestiya AN SSSR, Mekhanika tverdogo tela (Transactions of the Academy of Sciences of the USSR, Mechanics of Solids). 1970. No. 4. Pp. 150–162. (In Russian.)

4. Zhilin P.A., Konyushevskaya R.M., Palmov V.A., Chvartatsky R.V. On design of the stress-strain state of discharge chambers of Tokamak Panels. Leningrad, NIIEFA (Research Institute of Electro-physical apparatuses), P-OM-0550. 1982. Pp. 1–13. (In Russian.)

5. Zhilin P.A. Axisymmetric deformation of a cylindrical shell, supported by frames // Izvestiya AN SSSR, Mekhanika tverdogo tela (Transactions of the Academy of Sciences of the USSR, Mechanics of Solids). 1966, No. 5, pp. 139–142. (In Russian.)

A new formulation for the second law of thermodynamics for the case of thin surfaces

A new formulation for the second law of thermodynamics was proposed (1973) [1–4] by means of the combination of two Clausius–Duhem–Truesdell type inequalities. This formulation deals with a thin surface, each side of which has its own temperature and entropy. So, the formulation contains two entropies, two internal temperature fields, and two external temperature fields. Apart from the theory of shells this elaboration of the second law of thermodynamics is also useful for the solid-state physics when studying the influence of skin effects on properties of solids, as well as for the description of interfaces between different phases of a solid.

1. Zhilin P.A. Mechanics of Deformable Surfaces. The Danish Center for Appl. Math and Mech. Report N 89. 1975. P. 1–29.

2. Zhilin P.A. Mechanics of Deformable Cosserat Surfaces and Shell Theory. The Danish Center for Appl. Math and Mech. Annual report. 1975.

3. Zhilin P.A. Mechanics of deformable enriched surfaces // Transactions of the 9th Soviet conference on the theory of shells and plates. Leningrad, Sudostroenie. 1975. Pp. 48–54. (In Russian.)

4. Zhilin P.A. Mechanics of Deformable Directed Surfaces // Int. J. Solids Structures. 1976. Vol. 12. P. 635–648.

Generalization of the classical theory of symmetry of tensors

An important addition is made (1977) to the tensor algebra, namely the concept of oriented tensors, i.e. tensor objects which depend on orientation in both a three-dimensional space, and in its subspaces. The theory of symmetry [1, 2] is formulated for oriented tensors, and it generalises the classical theory of symmetry, which applies to the Euclidean tensors only. It was shown that the application of the classical theory, for example, to axial tensors, i.e. objects dependent on orientation in a 3D space, leads to wrong conclusions. The proposed theory is needed to obtain the constitutive equations for shells and other multipolar media, as well as when studying ionic crystals.

1. Zhilin P.A. General theory of constitutive equations in the linear theory of elastic shells // Izvestiya AN SSSR, Mekhanika tverdogo tela (Transactions of the Academy of Sciences of the USSR, Mechanics of Solids). 1978. No. 3. Pp. 190. (In Russian.)

2. Zhilin P.A. Basic equations of non-classical theory of shells // Dinamika i prochnost mashin (Dynamics and strength of machines.) Trudi LPI (Proceedings of Leningrad Polytechical Institute.) N 386. 1982. . 29–46. (In Russian.)

The general nonlinear theory of thermoelastic shells

The general nonlinear theory of thermoelastic shells is created (1975–1984). The way of its construction fundamentally differs from all known versions of shell theories and can be easily extended to any shell-like constructions and other objects of continuum mechanics. Its key feature is that it allows studying shell-like objects of a complex internal structure, i.e. when traditional methods of construction of the theory of shells are not applicable [1–11]. For shells of constant thickness, made of isotropic material, the new method gives results that are in accordance with those of the classical methods and perfectly coincide with the results of three-dimensional elasticity theory for the case of any external forces, including point loads.

1. Zhilin P.A. Two-dimensional continuum. Mathematical theory and physical interpretations // Izvestiya AN SSSR, Mekhanika tverdogo tela (Transactions of the Academy of Sciences of the USSR, Mechanics of Solids). 1972. N 6. Pp. 207–208. (In Russian.)

2. Zhilin P.A. Modern handling of the theory of shells // Izvestiya AN SSSR, Mekhanika tverdogo tela (Transactions of the Academy of Sciences of the USSR, Mechanics of Solids). 1974. N 4. (In Russian.)

3. Zhilin P.A. Mechanics of Deformable Surfaces. The Danish Center for Appl. Math and Mech. Report N 89. 1975. P. 1–29.

4. Zhilin P.A. Mechanics of Deformable Cosserat Surfaces and Shell Theory. The Danish Center for Appl. Math and Mech. Annual report. 1975.

5. Zhilin P.A. Mechanics of deformable enriched surfaces // Transactions of the 9th Soviet conference on the theory of shells and plates. Leningrad, Sudostroenie. 1975. Pp. 48–54. (In Russian.)

6. Zhilin P.A. Mechanics of Deformable Directed Surfaces // Int. J. Solids Structures. 1976. Vol. 12. P. 635–648.

7. Zhilin P.A. General theory of constitutive equations in the linear theory of elastic shells // Izvestiya AN SSSR, Mekhanika tverdogo tela (Transactions of the Academy of Sciences of the USSR, Mechanics of Solids). 1978. No. 3. Pp. 190. (In Russian.)

8. Zhilin P.A. A new method for the construction of theory of thin elastic shells // . . . Izvestiya AN SSSR, Mekhanika tverdogo tela (Transactions of the Academy of Sciences of the USSR, Mechanics of Solids). 1978. No. 3. (In Russian.)

9. Zhilin P.A. Direct construction of the theory of shells basing on physical principles // Izvestiya AN SSSR, Mekhanika tverdogo tela (Transactions of the Academy of Sciences of the USSR, Mechanics of Solids). 1980. No. 3. Pp. 179. (In Russian.)

10. Zhilin P.A. Basic equations of non-classical theory of shells // Dinamika i prochnost mashin (Dynamics and strength of machines.) Trudi LPI (Proceedings of Leningrad Polytechical Institute.) N 386. 1982. . 29–46. (In Russian.)

11. Altenbach H., Zhilin P.A. General theory of elastic simple shells // Advances in mechanics — Warszawa, Polska. 1988. N 4. P. 107–148. (In Russian).

Elimination of a paradox in the problem of bending deflection of a round plate

The exact analytical solution is given (1982) for the problem of final displacements of a round plate [1, 2]. The solution explains a well-known paradox which was described in handbooks and assumed that the deflection of a membrane, i.e. a plate with zero beam stiffness, was less than the deflection calculated with non-zero beam stiffness taken into account. (The problem considers a round plate with its edges fixed and loaded by transversal pressure, whose magnitude makes the application of the linear theory incorrect. The latter one overestimates the deflection approximately 25 times). Later the idea of works [1, 2] was used for calculation of an electrodynamic gate [3].

1. Zhilin P.A. Axisymmetric bending of a flexible circular plate under large displacements // Vichislitelnie metodi v mekhanike i upravlenii (Numerical methods in mechanics and control theory). Trudi LPI (Proceedings of Leningrad Polytechical Institute.) N 388. 1982. . 97–106. (In Russian.)

2. Zhilin P.A. Axisymmetrical bending of a circular plate under large displacements // Izvestiya AN SSSR, Mekhanika tverdogo tela (Transactions of the Academy of Sciences of the USSR, Mechanics of Solids). 1984. No. 3. Pp. 138–144. (In Russian.)

3. Venatovsky I.V., Zhilin P.A., Komyagin D.Yu. Inventor's certificate No. 1490663 with priority from 02.11.1987. (In Russian.)

Critical surveys

1. Zhilin P.A. The view on Poisson's and Kirchhoff's theories of plates in terms of modern theory of plates // Izvestia RAN. Mechanika Tverdogo Tela (Mechanics of Solids). 1992. N 3. P. 48–64. (In Russian).

2. Zhilin P.A. On the classical theory of plates and the Kelvin-Teit transformation // Izvestia RAN. Mechanika Tverdogo Tela (Mechanics of Solids). 1995. N 4. P. 133–140. (In Russian).

3. Altenbach H., Zhilin P.A. The Theory of Simple Elastic Shells // in Critical Review of The Theories of Plates and Shells and New Applications, ed. by H. Altenbach and R. Kienzler. Berlin, Springer. 2004. P. 1–12.

The theory of rods

The dynamic theory of thin spatially curvilinear rods and naturally twisted rods is developed (1987–2005). The proposed theory includes all known variants of theories of rods, but it has wider domain of application. A significant part of the work is devoted to the analysis of a series of classical problems, including those whose solutions demonstrate paradoxes. The results of the theory of rods and its applications are presented in the most complete way in work [1].

1. Zhilin P.A. Applied mechanics. Theory of thin elastic rods. Tutorial book. St. Petersburg State Polytechnical University. 2006. 98 p. (In Russian). To be published.

General nonlinear theory of rods and its applications to the solution of particular problems

Basing on method developed in the theory of shells, the general nonlinear theory of flexible rods is formulated (1987), where all the basic types of deformation: bending, torsion, tension, transversal shear are taken into account. Use of the rotation (turn) tensor allowed to write down the equations in a compact form, convenient for the mathematical analysis. In contradistinction to previous theories, the proposed theory describes the experimentally discovered Pointing effect (the contraction of a rod under torsion). The developed theory was applied to analyse the series of particular problems [2, 3]. A new method [4–6] was suggested (2005) for the construction of elastic tensors, and their structure has been determined. In this work the new theory of symmetry of tensors, determined in the space with two independent orientations, is essentially used. All elastic constants were found for plane curvilinear rods.

1. Goloskokov D.P., Zhilin .. General nonlinear theory of elastic rods with application to the description of the Pointing effect // Deposited in VINITI No. 1912-V87. Dep. 20 p. (In Russian.)

2. Zhilin P.A., Tovstik T.P. Rotation of a rigid body based on an inertial rod // Mechanics and Control. Proc. of St. Petersburg State Technical University. 1995. N 458. P. 78–83. (In Russian).

3. Zhilin P.A., Sergeyev A.D., Tovstik T.P. Nonlinear theory of rods and its application // Proc. of XXIV Summer School - Seminar “Nonlinear Oscillations in Mechanical Systems”. St. Petersburg. 1997. P. 313–337. (In Russian).

4. Zhilin P.A. Theory of thin elastic rods //Lecture at XXXIII Summer School - Conference “Advanced Problems in Mechanics”. St. Petersburg, Russia. 2005. Current book. Vol. 1. (In Russian).

5. Zhilin P.A. Nonlinear Theory of Thin Rods // Lecture at XXXIII Summer School – Conference “Advanced Problems in Mechanics”. St. Petersburg, Russia. 2005. Current book. Vol. 2.

6. Zhilin P.A. Applied mechanics. Theory of thin elastic rods. Tutorial book. St. Petersburg State Polytechnical University. 2006. 98 p. (In Russian). To be published.

The Euler elastica

The famous Euler elastica [1–5] was considered (1997–2005) and it was shown that apart from the known static equilibrium configurations there exist also dynamic equilibrium configurations. In the latter case, the form of elastic curve remains the same, and the bent rod rotates about the vertical axe. The energy of deformation does not change in this motion. Note that we do not speak about the rigid motion of a rod, since the clamped end of the rod remains fixed. This means that the curvilinear equilibrium configuration in the Euler elastica is unstable, contrary to the common point of view. On the other hand, this conclusion is not confirmed by experiments. Thus there appears a paradox requiring its explanation.

1. Zhilin P.A., Sergeyev A.D., Tovstik T.P. Nonlinear theory of rods and its application // Proc. of XXIV Summer School - Seminar “Nonlinear Oscillations in Mechanical Systems”. St. Petersburg. 1997. P. 313–337. (In Russian).

2. Zhilin P.A. Dynamic Forms of Equilibrium Bar Compressed by a Dead Force // Proc. of 1st Int. Conf. Control of Oscillations and Chaos. Vol. 3. 1997. P. 399–402.

3. Zhilin P.A. Theory of thin elastic rods //Lecture at XXXIII Summer School - Conference “Advanced Problems in Mechanics”. St. Petersburg, Russia. 2005. Current book. Vol. 1. (In Russian).

4. Zhilin P.A. Nonlinear Theory of Thin Rods // Lecture at XXXIII Summer School – Conference “Advanced Problems in Mechanics”. St. Petersburg, Russia. 2005. Current book. Vol. 2.

5. Zhilin P.A. Applied mechanics. Theory of thin elastic rods. Tutorial book. St. Petersburg State Polytechnical University. 2006. 98 p. (In Russian). To be published.

Nikolai paradox

The Nikolai paradox [1–7] is analysed (1993-2005). The paradox appears when a rod is subjected to the torsion by means of the torque applied to its end. The experiment shows that the torsion torque stabilises the rod, which is in the major contradiction with the theory. It is shown [6], that one may avoid the mentioned paradox if to choose a special constitutive equation for the torque. The torque has to depend in a special way on the rotational velocity. This dependence is not related to the existence (or absence) of the internal friction in the rod.

1. Zhilin P.A., Sergeyev A.D. Twisting of an elastic cantilever rod by a torque subjected at a free end. St. Petersburg State Technical University. 1993. 32 p. (In Russian).

2. Zhilin P.A., Sergeyev A.D. Experimental investigation of the stability of a cantilever rod under torsion deforming // Mechanics and Control. Proc. of St. Petersburg State Technical University. 1993. N 446. P. 174–175. (In Russian).

3. Zhilin P.A., Sergeyev A.D. Equilibrium and stability of a thin rod subjected to a conservative moment // Mechanics and Control. Proc. of St. Petersburg State Technical University. 1994. N 448. P. 47–56. (In Russian).

4. Zhilin P.A., Sergeyev A.D., Tovstik T.P. Nonlinear theory of rods and its application // Proc. of XXIV Summer School - Seminar “Nonlinear Oscillations in Mechanical Systems”. St. Petersburg. 1997. P. 313–337. (In Russian).

5. Zhilin P.A. Theory of thin elastic rods //Lecture at XXXIII Summer School - Conference "Advanced Problems in Mechanics". St. Petersburg, Russia. 2005. Current book. Vol. 1. (In Russian).

6. Zhilin P.A. Nonlinear Theory of Thin Rods // Lecture at XXXIII Summer School – Conference "Advanced Problems in Mechanics". St. Petersburg, Russia. 2005. Current book. Vol. 2.

7. Zhilin P.A. Applied mechanics. Theory of thin elastic rods. Tutorial book. St. Petersburg State Polytechnical University. 2006. 98 p. (In Russian). To be published.

The development of mathematical methods

An approach [1] is suggested (1995), which allows to analyse the stability of motion in the presence of spinor motions described by means of rotation (turn) tensor. The problem is that the rotation tensors are not elements of a linear space (unlike the displacement vectors). Thus the equations in variations have to be written down as a chain of equations, whose right parts depend on the previous variations in a nonlinear way. However, the obtained chain of equations allows for the exact separation of variables, i.e. the separation of the time variable.

1. Zhilin P.A. Spin motions and stability of equilibrium configurations of thin elastic rods // Mechanics and Control. Proc. of St. Petersburg State Technical University. 1995. N 458. P. 56–73. (In Russian).

Dynamics of rigid bodies

It was the first time when the dynamics of rigid bodies was formulated in terms of the direct tensor calculus. The new mathematical technique is developed for the description of spinor motions. This technique is based on the use of the rotation (turn) tensor and related concepts. The new results in the dynamics of rigid bodies are mostly presented in the following works:

1. Zhilin P.A. Theoretical mechanics. Tutorial book. St. Petersburg State Polytechnical University. 2001. 146 p. (In Russian).

2. Zhilin P.A. Vectors and second-rank tensors in three-dimensional space. St. Petersburg: Nestor. 2001. 276 p.

3. Zhilin P.A. Theoretical mechanics. Fundamental laws of mechanics. Tutorial book. St. Petersburg State Polytechnical University. 2003. 340 p. (In Russian).

Development of mathematical methods

The general investigation of the rotation (turn) tensor is given (1992) in works [1, 7, 8], where a new proof of the kinematic equation of Euler is obtained. The old correct proof of the kinematic equation one could find in works by L. Euler and in old text-books on theoretical mechanics, but it was very tedious. In a well-known course by T. Levi-Civitta and U. Amaldi (1922) a new compact proof was suggested, but it was erroneous. Later this proof was widely distributed and repeated in almost all modern courses on theoretical mechanics, with exception of the book by G.K. Suslov. In work [1] the proof of a new theorem on the composition of angular velocities, different from those cited in traditional text-books, is proposed.

The new equation [1, 4–8] is obtained (1992), relating the left angular velocity with the derivative of the rotation vector. This equation is necessary to define the concept of a potential torque. Apart from that, it is very useful when solving numerically the problems of dynamics of rigid bodies, since then there is no need to introduce neither systems of angles, nor systems of parameters of the Klein-Hamilton type.

A new theorem [2, 3, 7, 8] on the representation of the rotation (turn) tensor in the form of a composition of turns about arbitrary fixed axes, is proved (1995). All previously known representations of the rotation (turn) tensors, (or, saying more precisely, of its matrix analogues) via Euler angles, Brayant angles, plane angles, ship angles etc., are particular cases of a general theorem, whose role, however, is not only a simple generalisation of these cases. The most important thing is that making a traditional choice of any system of angles, does not matter which one, we choose previously the axes. We describe the (unknown) rotation of a body under consideration in terms of turns about these axes. If this choice is made in an ineffectual way, and if it is difficult to make an appropriate choice, the chances to integrate or even to analyse qualitatively the resulting system of equations are very poor. Moreover, even in those cases when it is possible to integrate the system, often the obtained solution is not of big practical use, since this solution will contain poles or indeterminacy of the type zero divided by zero. As a result, the numerical solution, obtained with the help of computers, already after the first pole or indeterminacy becomes very distorted. The advantage and the purpose of the theorem under discussion is the fact that it allows to consider the axes of rotation as principal variables and to determine them in the process of the problem solution. As a result, one can obtain the simplest (among all possible forms) solutions.

An approach [4–6] is proposed (1997), which allows to analyse the stability of motion in the presence of spinor rotations described by the turn tensor. The method of perturbations for the group of proper orthogonal tensors is developed.

1. Zhilin P.A. The turn-tensor in kinematics of a rigid body // Mechanics and Control. Proc. of St. Petersburg State Technical University. 1992. N 443. P. 100–121. (In Russian).

2. Zhilin P.A. A New Approach to the Analysis of Euler-Poinsot problem // ZAMM. Z. angew. Math. Mech. **75**. (1995) S 1. P. 133–134.

3. Zhilin P.A. A New Approach to the Analysis of Free Rotations of Rigid Bodies // ZAMM. Z. angew. Math. Mech. **76**. (1996) N 4. P. 187–204.

4. Zhilin P.A. Dynamics and stability of equilibrium positions of a rigid body on an elastic foundation //Proc. of XXIV Summer School - Seminar “Nonlinear Oscillations in Mechanical Systems”. St. Petersburg. 1997. P. 90–122. (In Russian).

5. Zhilin P.A. A General Model of Rigid Body Oscillator // Proc. of the XXV-XXIV Summer Schools "Nonlinear Oscillations in Mechanical Systems". Vol. 1. St. Petersburg. 1998. P. 288–314.

6. Zhilin P.A. Rigid body oscillator: a general model and some results // Acta Mechanica. Vol. 142. (2000) P. 169–193.

7. Zhilin P.A. Vectors and second-rank tensors in three-dimensional space. St. Petersburg: Nestor. 2001. 276 p.

8. Zhilin P.A. Theoretical mechanics. Fundamental laws of mechanics. Tutorial book. St. Petersburg State Polytechnical University. 2003. 340 p. (In Russian).

New solutions of classical problems

A new solution [1, 2] is obtained (1995) for the classical problem of the free rotation of a rigid body about a fixed centre of mass (case of Euler). It is shown that for each inertia tensor all the domain of initial values is divided in two subdomains. It is known that there is no such a system of parameters, which would allow to cover all the domain of initial values by unique map without poles. This fact is confirmed in the work [2], where in each subdomain and at the boundary between them the body rotates about different axes, depending only on the initial values. Stable rotations of the body correspond to the interior points of the subdomains mentioned above, and unstable rotations — to the boundary points. When constructing the solution, the theorem on the representation of the rotation (turn) tensor, described above, plays an essential role. Finally, all characteristics to be found can be expressed via one function, determined by a rapidly convergent series of a quite simple form. For this reason, no problem appears in simulations. The propriety of the determination of axes, about which the body rotates, manifests in the fact that the velocities of precession and proper rotation have a constant sign. Remind that in previously known solutions only the sign of the precession velocity is constant, i.e. in these solutions only one axe of turns is correctly guessed. It follows from the solution [2], that formally stable solutions, however, may be unstable in practice, if a certain parameter is small enough (zero value of the parameter corresponds to the boundary between subdomains). In this case the body may jump from one stable rotational regime to another one under action of arbitrarily small and short loads (a percussion with a small meteorite).

A new solution [3, 4] for the classical problem of the rotation of a rigid body with transversally isotropic inertia tensor is obtained (1996, 2003) in a homogeneous gravity field (case of Lagrange). The solution of this problem from the formal mathematical point of view is known very long ago, and one can find it in many monographs and text-books. However, it is difficult to make a clear physical interpretation of this solution, and some simple types of motion are described by it in an unjustifiably sophisticated way. In the case of a rapidly rotating gyroscope there was obtained practically an exact solution in elementary functions. It was shown [4] that the expression for the precession velocity, found using the elementary theory of gyroscopes, gives an error in the principal term.

It is found (2003), in the frame of the dynamics of rigid bodies, the explanation of the fact that the velocity of the rotation of the Earth is not constant, and the axe of the

Earth is slightly oscillating [5]. Usually this fact is explained by the argument that one cannot consider the Earth as an absolutely rigid body. However, if the direction of the dynamic spin slightly differs from the direction of the earth axe, the earth axe will make precession about the vector of the dynamic spin, and, consequently, the angle between the axe of the Earth and the plane of ecliptics will slightly change. In this case the alternation of day and night on the Earth will be determined not by the proper rotation of the Earth about its axe, but by the precession of the axe.

1. Zhilin P.A. A New Approach to the Analysis of Euler-Poinsot problem // ZAMM. Z. angew. Math. Mech. **75**. (1995) S 1. P. 133–134.

2. Zhilin P.A. A New Approach to the Analysis of Free Rotations of Rigid Bodies // ZAMM. Z. angew. Math. Mech. **76**. (1996) N 4. P. 187–204.

3. Zhilin P.A. Rotations of Rigid Body with Small Angles of Nutation // ZAMM. Z. angew. Math. Mech. **76**. (1996) S 2. P. 711–712.

4. Zhilin P.A. Rotation of a rigid body with a fixed point: the Lagrange case //Lecture at XXXI Summer School - Conference "Advanced Problems in Mechanics". St. Petersburg, Russia. 2003. Current book. Vol. 1. (In Russian).

5. Zhilin P.A. Theoretical mechanics. Fundamental laws of mechanics. Tutorial book. St. Petersburg State Polytechnical University. 2003. 340 p. (In Russian).

New models in the frame of the dynamics of rigid bodies

We know the role which is played by a usual oscillator in the Newtonian mechanics. In the Eulerian mechanics, the analogous role is played by a rigid body on an elastic foundation. This system can be named *a rigid body oscillator*. The last one is necessary when constructing the dynamics of multipolar media, but in its general case it is not investigated neither even described in the literature. Of course, its particular cases were considered, for instance, in the analysis of the nuclear magnetic resonance, and also in many applied works, but for infinitesimal angles of rotation. A new statement of the problem of the dynamics of a rigid body on a nonlinear elastic foundation [1, 3, 6] is proposed (1997). The general definition of the potential torque is introduced. Some examples of problem solutions are given.

For the first time (1997) the mathematical statement for the problem of a two-rotor gyrostate on an elastic foundation is given [2, 4, 5]. The elastic foundation is determined by setting of the strain energy as a scalar function of the rotation vector. Finally, the problem is reduced to the integration of a system of nonlinear differential equations having a simple structure but a complex nonlinearity. The difference of these equations from those traditionally used in the dynamics of rigid bodies is that when writing them down it is not necessary to introduce any artificial parameters of the type of Eulerian angles or Cayley-Hamilton parameters. The solutions of concrete problems are considered. A new method of integration of basic equations is described in application to a particular case. The solutions is obtained in quadratures for the isotropic nonlinear elastic foundation.

The model of a rigid body is generalised (2003) for the case of a body consisting not of the mass points, but of the point-bodies of general kind [7]. There was considered a model of a quasi-rigid body, consisting of the rotating particles, with distances between them remaining constant in the process of motion.

1. Zhilin P.A. Dynamics and stability of equilibrium positions of a rigid body on an elastic foundation //Proc. of XXIV Summer School - Seminar "Nonlinear Oscillations in Mechanical Systems". St. Petersburg. 1997. P. 90–122. (In Russian).

2. Zhilin P.A., Sorokin S.A. Multi-rotor gyrostat on a nonlinear elastic foundation // IPME RAS. Preprint N 140. 1997. 83 p. (In Russian).

3. Zhilin P.A. A General Model of Rigid Body Oscillator // Proc. of the XXV-XXIV Summer Schools "Nonlinear Oscillations in Mechanical Systems". Vol. 1. St. Petersburg. 1998. P. 288–314.

4. Zhilin P.A., Sorokin S.A. The Motion of Gyrostat on Nonlinear Elastic Foundation // ZAMM. Z. Angew. Math. Mech. **78**. (1998) S 2. P. 837–838.

5. Zhilin P.A. Dynamics of the two rotors gyrostat on a nonlinear elastic foundation // ZAMM. Z. angew. Math. Mech. **79**. (1999) S 2. P. 399–400.

6. Zhilin P.A. Rigid body oscillator: a general model and some results // Acta Mechanica. Vol. 142. (2000) P. 169–193.

7. Zhilin P.A. Theoretical mechanics. Fundamental laws of mechanics. Tutorial book. St. Petersburg State Polytechnical University. 2003. 340 p. (In Russian).

Dynamics of a rigid body on an inertial elastic foundation

The problems of construction of high-speed centrifuges, with rotational velocities 120 000 – 200 000 revolutions per minute, required the development of more sophisticated mechanical models. As such a model it is chosen a rigid body on an elastic foundation. The parameters of the rotor and of the elastic foundation do not allow to consider the elastic foundation as inertialess. There was proposed (1995) a method [1, 2], allowing to reduce the problem to the solution of a relatively simple integro-differential equation.

1. Zhilin P.A., Tovstik T.P. Rotation of a rigid body based on an inertial rod // Mechanics and Control. Proc. of St. Petersburg State Technical University. 1995. N 458. P. 78–83. (In Russian).

2. Ivanova E.A., Zhilin P.A. Non-stationary regime of the motion of a rigid body on an elastic plate // Proc. of XXIX Summer School – Conference "Advanced Problems in Mechanics". St. Petersburg. 2002. P. 357–363.

The Coulomb law of friction and paradoxes of Painlevé

The application of the Coulomb law has its own specifics related to the non-uniqueness of the solution of the dynamics problems. It was shown (1993), that the Painlevé paradoxes appear because of a priori assumptions on the character of motion and the character of the forces needed to induce this motion. The correct statement of the problem requires either to determine the forces by the given motion, or to determine the motion by the given forces [1, 2].

1. Zhilin P.A., Zhilina O.P. On the Coulomb's laws of friction and the Painlevé paradoxes // Mechanics and Control. Proc. of St. Petersburg State Technical University. 1993. N 446. P. 52–81. (In Russian).

2. Wiercigroch M., Zhilin P.A. On the Painlevé Paradoxes // Proc. of the XXVII Summer School "Nonlinear Oscillations in Mechanical Systems". St. Petersburg. 2000. P. 1–22.

The fundamental laws of mechanics

There were suggested (1994) the formulations of basic principles and laws of the Eulerian mechanics [1–5] with an explicit introduction of spinor motions. All the laws are formulated for the open bodies, i.e. bodies of a variable content, which appears to be extremely important when describing the interaction of macrobodies with electromagnetic fields. Apart from that, in these formulations the concept of a body itself is also changed, and now the body may contain not only particles, but also the fields. Namely, the latter ones make necessary to consider bodies of variable content. The importance of spinor motions, in particular, is determined by the fact that the true magnetism can be defined only via the spinor motions, contrary to the induced magnetism, caused by Foucault (eddy) currents, i.e. by translational motions.

A new basic object — point-body [1–5] is introduced into consideration (1994). It is assumed that the point-body occupies zero volume, and its motion is described completely by means of its radius-vector and its rotation (turn) tensor. It is postulated that the kinetic energy of a point-body is a quadratic form of its translational and angular velocities, and its momentum and proper kinetic moment (dynamic spin) are defined as partial derivatives of the kinetic energy with respect to the vector of translational velocity and the vector of angular velocity, respectively. It was considered (2003) the model of a point-body [5], whose structure is determined by three parameters: mass, inertia moment, and an additional parameter q, conventionally named *charge*, which never appeared in particles used in classical mechanics. It is shown that the motion of this particle by inertia in a void space has a spiral trajectory, and for some initial conditions — a circular trajectory. Thus it is shown that in an inertial frame reference the motion of an isolated particle (point-body) by inertia has not to follow necessarily a linear path.

There was developed (1994) a concept of actions [1–5]. This concept is based on an axiom which supplements the Galileo's Principle of Inertia, generalising it to the bodies of general kind. This axiom states that in an inertial system of reference an isolated closed body moves in such a way that its momentum and kinetic moment remain invariable. Further, the forces and torques are introduced into consideration, and the force acting upon a closed body is defined as a cause for the change of the momentum of this body, and the torque, acting upon a closed body — as a cause of the change of the kinetic moment. The couple of vectors — force vector and the couple vector — are called *action*.

The concept of the internal energy of a body, consisting of point-bodies of general kind [1–5], was developed (1994); the axioms for the internal energy to be satisfied are formulated. The principally new idea is to distinguish the additivity by mass and additivity by bodies.

The kinetic energy of a body is additive by its mass. At the same time, the internal energy of a body is additive by sub-bodies of which the body under consideration consists of, but, generally speaking, it is not an additive function of mass. In the Cayley problem, the paradox, related to the loss of energy, is resolved [5].

Basic concepts of thermodynamics [4, 5]: internal energy, temperature, and entropy are introduced (2002) on elementary examples of mechanics of discrete systems. The definition of the temperature concepts and entropy are given by means of purely mechanical arguments, based on the use of a special mathematical formulation of the energy balance.

1. Zhilin P.A. Main structures and laws of rational mechanics // Proc. of the 1st Soviet Union Meeting for Heads of Departments of Theoretical Mechanics. St. Petersburg: VIKI. 1994. P. 23–45. (In Russian).

2. Zhilin P.A. Basic concepts and fundamental laws of rational mechanics // Proc. of XXII Summer School - Seminar "Nonlinear Oscillations in Mechanical Systems". St. Petersburg. 1995. P. 10–36. (In Russian).

3. Zhilin P.A. Theoretical mechanics. Tutorial book. St. Petersburg State Polytechnical University. 2001. 146 p. (In Russian).

4. Zhilin P.A. Basic postulates of the Eulerian mechanics // Proc. of XXIX Summer School - Conference "Advanced Problems in Mechanics". St. Petersburg. 2002. P. 641–675. (In Russian)

5. Zhilin P.A. Theoretical mechanics. Fundamental laws of mechanics. Tutorial book. St. Petersburg State Polytechnical University. 2003. 340 p. (In Russian).

Electrodynamics

It is shown [1, 2], that Maxwell equations are invariant with respect to the Galilean transformation, i.e. the principle of relativity by Galileo is valid for them (we distinguish transformations of frames of reference and of co-ordinate systems). The complete group of linear transformations, with respect to which the Maxwell equations are covariant, is found, and it is demonstrated that Lorentz transformations present quite a particular case of the complete group.

The role, which electromechanical analogies play in the analytical mechanics of mass points, is well-known. For the electrodynamic equations, such analogies in the modern theoretical physics are not only unknown, but are even denied. In work [3], mathematically rigorous mechanical interpretation of the Maxwell equations is given, and it is shown that they are completely identical to the equations of oscillations of a non-compressible elastic medium. Thus it follows that in the Maxwell equations there is an infinite velocity of the propagation of extension waves, which is in the explicit contradiction with special relativity theory. In other words, electrodynamics and special relativity theory are incompatible. These analogies were established by Maxwell himself for the absence of charges, and in [3] they are proved for the general case.

The modified Maxwell equations are proposed [3–5]. In the modified theory, all the waves propagate with finite velocities, one of them has to be greater than the light velocity in vacuum. If this to consider the limit case, when this velocity tends to the infinity, the modified equations give the Maxwell equations in the limit. The waves with the "superlight" velocity are longitudinal. One cannot eliminate the possibility that these waves describe the phenomenon of radiation propagating with the velocity greater than the light velocity, which is claimed to be experimentally observed by some astronomers.

It is established [3–5] that in terms of this theory electrostatic states present hyperlight waves and are realised far from the wave front.

It is shown [3], that neither classical, nor modified Maxwell equations cannot describe correctly the interaction between the electrons and the nucleus of the atom. The way to solve this problem is indicated.

It is shown [6], that the mathematical description of an elastic continuum of two-spin particles of a special type is reduced to the classical Maxwell equations. The mechanical analogy proposed above allows to state unambigously that the vector of electric field is axial, and the vector of magnetic field is polar.

1. Zhilin P.A. Galileo's equivalence principle and Maxwell's equations. // St. Petersburg State Technical University. 1993. 40 p. (In Russian).

2. Zhilin P.A. Galileo's equivalence principle and Maxwell's equations // Mechanics and Control. Proc. of St. Petersburg State Technical University. 1994. N 448. P. 3–38. (In Russian).

3. Zhilin P.A. Reality and mechanics // Proc. of XXIII Summer School - Seminar "Nonlinear Oscillations in Mechanical Systems". St. Petersburg. 1996. P. 6–49. (In Russian).

4. Zhilin P.A. Classical and Modified Electrodynamics // Proc. of Int. Conf. "New Ideas in Natural Sciences". St. Petersburg, Russia. June 17–22, 1996. Part I – Physics. P. 73–82.

5. Zhilin P.A. Classical and modified electrodynamics // Proc. of the IV International Conference "Problems of Space, Time, and Motion" dedicated to the 350th anniversary of Leibniz. St. Petersburg. 1997. Vol. 2. P. 29–42. (In Russian)

6. Zhilin P.A. The Main Direction of the Development of Mechanics for XXI century // Lecture prepared for presentation at XXVIII Summer School – Conference "Advanced Problems in Mechanics". St. Petersburg, Russia. 2000. Current book. Vol. 2.

Quantum mechanics

At the end of the XIX century Lord Kelvin described the structure of an aether responsible, in his opinion, for the true (non-induced) magnetism, consisting of rotating particles. A kind of specific Kelvin medium (aether) is considered: the particles of this medium cannot perform translational motion, but have spinor motions. Lord Kelvin could not write the mathematical equations of such motion, because the formulation of the rotation tensor, a carrier of a spinor motion, was not discovered at the time. In work [1, 2] basic equations of this particular Kelvin medium are obtained, and it is shown that they present a certain combination of the equations of Klein-Gordon and Schrödinger. At small rotational velocities of particles, the equations of this Kelvin medium are reduced to the equations of Klein-Gordon, and at large velocities — to the Schrödinger equation. It is very significant that both equations lie in the basis of quantum mechanics.

1. Zhilin P.A. Reality and mechanics // Proc. of XXIII Summer School - Seminar "Nonlinear Oscillations in Mechanical Systems". St. Petersburg. 1996. P. 6–49. (In Russian).

2. Zhilin P.A. Classical and Modified Electrodynamics // Proc. of Int. Conf. "New Ideas in Natural Sciences". St. Petersburg, Russia. June 17–22, 1996. Part I – Physics. P. 73–82.

General theory of inelastic media

A general approach [1–6] for the construction of the theory of inelastic media is proposed (2001–2005). The main attention is given to the clear introduction of basic concepts: strain measures, internal energy, temperature, and chemical potential. Polar and non-polar media are considered. The originality of the suggested approach is in the following. The spatial description is used. The fundamental laws are formulated for the open systems. A new handling of the equation of the balance of energy is given, where the entropy and the chemical potential are introduced by means of purely mechanical arguments. The internal energy is given in a form, at the same time applicable for gaseous, liquid, and solid bodies. Phase transitions in the medium are described without introducing any supplementary conditions; solid-solid phase transition can also be described in these terms. The materials under consideration have a finite tensile strength; this means that the constitutive equations satisfy to the condition of the strong ellipticity.

1. Zhilin P.A. Basic equations of the theory of non-elastic media // Proc. of the XXVIII Summer School "Actual Problems in Mechanics". St. Petersburg. 2001. P. 14–58. (In Russian).

2. Zhilin P.A. Phase Transitions and General Theory of Elasto-Plastic Bodies // Proc. of XXIX Summer School – Conference "Advanced Problems in Mechanics". St. Petersburg. 2002. P. 36–48.

3. Altenbach H., Naumenko K., Zhilin P. A micro-polar theory for binary media with application to phase-transitional flow of fiber suspensions // Continuum Mechanics and Thermodynamics. 2003. Vol. 15. N 6. P. 539–570.

4. Altenbach H., Naumenko K., Zhilin P.A. A micro-polar theory for binary media with application to flow of fiber suspensions // Proc. of XXX Summer School – Conference "Advanced Problems in Mechanics". St. Petersburg. 2003. P. 39–62.

5. Zhilin P.A. Mathematical theory of non-elastic media // Uspehi mechaniki (Advances in mechanics). Vol. 2. N 4. 2003. P. 3–36. (In RUssian.)

6. Zhilin P.A. On the general theory of non-elastic media // Mechanics of materials and strength of constructions. Proc. of St. Petersburg State Polytechnical University. N 489. 2004. P. 8–27. (In Russian).

Spatial description of the kinematics of continuum

When constructing the general theory of inelastic media there was used (2001) so called *spatial description* [1–4], where a certain fixed domain of a frame of reference contains different medium particles in different moments of time. Due to the use of the spatial description there was constructed a theory, where the concept of the smooth differential manifold is not used. Before such theories were developed only for fluids. For the first time such a theory is built for solids, where the stress deviator is non-zero. For the first time, the spatial description is applied to a medium consisting of particles with rotational degrees of freedom. A new definition of a material derivative, containing only objective operators, is given. This definition, including when using a moving co-ordinate system, does not contradict to the Galileo's Principle of Inertia [2]. It is shown that for the spatial description one can apply standard methods of the introduction of the stress tensor and other similar quantities [1]. The dynamic equations of the medium obtained basing upon the fundamental laws, formulated for the open systems. An error, which presents in the literature, appearing when integrating the differential equation expressing the law of conservation of particles, is eliminated.

1. Zhilin P.A. Basic equations of the theory of non-elastic media // Proc. of the XXVIII Summer School "Actual Problems in Mechanics". St. Petersburg. 2001. P. 14–58. (In Russian).

2. Zhilin P.A. Phase Transitions and General Theory of Elasto-Plastic Bodies // Proc. of XXIX Summer School – Conference "Advanced Problems in Mechanics". St. Petersburg. 2002. P. 36–48.

3. Zhilin P.A. Mathematical theory of non-elastic media // Uspehi mechaniki (Advances in mechanics). Vol. 2. N 4. 2003. P. 3–36. (In Russian.)

4. Zhilin P.A. On the general theory of non-elastic media // Mechanics of materials and strength of constructions. Proc. of St. Petersburg State Polytechnical University. N 489. 2004. P. 8–27. (In Russian).

Theory of strains

Usually in the nonlinear theory of elasticity the theory of deformations is based only on geometrical reasons, thus a lot of different strain tensors are considered. It is usually assumed that all of these tensors can be used with identical success. However, this is not correct. It is shown (2001), that the dissipative inequality imposes such restrictions on free energy [1, 2], which at use of Almansi measure of strain appear equivalent to the statement, that the considered material is isotropic. In other words it is shown, that for anisotropic materials free energy cannot be a function of Almansi measure of strain. The definition of the strain measure is given on the base of the equation of balance of energy and the dissipative inequality. It is shown, that according to the given definition the strain measure should be an unimodular tensor.

1. Zhilin P.A. Basic equations of the theory of non-elastic media // Proc. of the XXVIII Summer School "Actual Problems in Mechanics". St. Petersburg. 2001. P. 14–58. (In Russian).

2. Zhilin P.A. Phase Transitions and General Theory of Elasto-Plastic Bodies // Proc. of XXIX Summer School – Conference "Advanced Problems in Mechanics". St. Petersburg. 2002. P. 36–48.

Equation of mass balance and equation of particles balance

Two independent functions of state are introduced: density of particles and mass density (2002) [1–3]. Such division is important, for example, when the material tends to a fragmentation, as in this case the weight is preserved, but the number of particles changes. Permeability of bodies is determined by the density of particles, and internal interactions are connected with the mass density. Introduction of the function of distribution of particles, in essence, removes the border between discrete and continuous media. Two independent equations are formulated: the equation of mass balance and the equation of balance of particles. A function determining the speed of new particles creation appears in the equation of particles balance; this function in its physical sense can be identified with the chemical potential. The equation of energy balance also contains terms which describe formation of new particles or fragmentation of original particles.

1. Zhilin P.A. Phase Transitions and General Theory of Elasto-Plastic Bodies // Proc. of XXIX Summer School – Conference "Advanced Problems in Mechanics". St. Petersburg. 2002. P. 36–48.

2. Zhilin P.A. Mathematical theory of non-elastic media // Uspehi mechaniki (Advances in mechanics). Vol. 2. N 4. 2003. P. 3–36. (In Russian.)

3. Zhilin P.A. On the general theory of non-elastic media // Mechanics of materials and strength of constructions. Proc. of St. Petersburg State Polytechnical University. N 489. 2004. P. 8–27. (In Russian).

Temperature, entropy and chemical potential

Characteristics of state, corresponding to temperature, entropy, and chemical potential are obtained [1–4] from pure mechanical reasons, by means of special mathematical formulation of the energy balance equation (2001), obtained by separation of the stress tensors in elastic and dissipative components. The second law of thermodynamics gives additional limitations for the introduced characteristics, and this completes their formal definition. The reduced equation of energy balance is obtained in the terms of free energy. The main purpose of this equation is to determine the arguments on which the free energy depends. It is shown that to define first the internal energy, and then the entropy and chemical potential, is impossible. All these quantities should be introduced simultaneously. To set the relations between the internal energy, entropy, chemical potential, pressure, etc., the reduced equation of energy balance is used. It is shown that the free energy is a function of temperature, density of particles, and strain measures, where all these arguments are independent. The Cauchy-Green's relations relating entropy, chemical potential and tensors of elastic stresses with temperature, density of particles and measures of deformation are obtained. Hence the concrete definition of the constitutive equations requires the setting of the free energy only.

The equations characterizing role of entropy and chemical potential in formation of internal energy are obtained. Constitutive equations for the vector of energy flux [3] are offered. In a particular case these equations give the analogue of the Fourier-Stocks law.

1. Zhilin P.A. Basic equations of the theory of non-elastic media // Proc. of the XXVIII Summer School "Actual Problems in Mechanics". St. Petersburg. 2001. P. 14–58. (In Russian).

2. Zhilin P.A. Phase Transitions and General Theory of Elasto-Plastic Bodies // Proc. of XXIX Summer School – Conference "Advanced Problems in Mechanics". St. Petersburg. 2002. P. 36–48.

3. Zhilin P.A. Mathematical theory of non-elastic media // Uspehi mechaniki (Advances in mechanics). Vol. 2. N 4. 2003. P. 3–36. (In Russian.)

4. Zhilin P.A. On the general theory of non-elastic media // Mechanics of materials and strength of constructions. Proc. of St. Petersburg State Polytechnical University. N 489. 2004. P. 8–27. (In Russian.)

Theory of consolidating granular media

The general theory of granular media with particles able to join (consolidate) is developed (2001) [1, 2]. The particles possess translational and rotational degrees of freedom. For isotropic material with small displacements and isothermal strains the theory of consolidating granular media is developed in a closed form [1].

It is shown that the assumption that the tensor of viscous stresses depends on velocity, leads either to failure of dissipative inequality, or to failure of hyperbolicity [1]. Hence this assumption is unacceptable. Instead of the tensor of viscous stresses, which is frequently used in literature, the antisymmetric stress tensor is introduced [1]. For this tensor the Coulomb friction law is used. For the couple stress tensor the viscous friction law is used, and this tensor is assumed to be antisymmetric.

1. Zhilin P.A. Basic equations of the theory of non-elastic media // Proc. of the XXVIII Summer School "Actual Problems in Mechanics". St. Petersburg. 2001. P. 14–58. (In Russian).

2. Zhilin P.A. Phase Transitions and General Theory of Elasto-Plastic Bodies // Proc. of XXIX Summer School – Conference "Advanced Problems in Mechanics". St. Petersburg. 2002. P. 36–48.

Phase Transitions and General Theory of Elasto-Plastic Bodies

The new theory of elasto-plastic bodies is developed (2002). The theory is based on the description of the nonelastic properties by the phase transitions in the materials [1–3].

The definition of the phase transition is given in the following way. Two material characteristics are related to the density of material: solid fraction, defined as multiplication of number of particles in a unit volume on the particle volume, and porosity (void fraction), defined as unit minus solid fraction. A solid has several stable states, corresponding to different values of solid fraction. Transition from one stable state to another is a typical phase transition. The constitutive equation describing the solid fraction change near the phase transition point is suggested.

The constitutive equation for elastic pressure is proposed [1]. This equation describes well not only gases and liquids, but also solids with phase transitions. The limited tensile strength is taken into account. The difference between solids and liquids mainly is in their reaction on the shape change. This reaction can be described only if the stress tensor deviator is taken into account. For the classical approach the deviator of the elastic stress tensor, which is independent of velocities by definition, is ignored for description of inelastic properties of materials. For solids this is unacceptable. One of the problems of the theory is the definition of the internal energy structure in a way that would make possible the existence of several solid phases. The constitutive equations for the stress tensor deviator are suggested [1], where the shear modulus depends on the state parameters (temperature, mass density, deformation).

1. Zhilin P.A. Phase Transitions and General Theory of Elasto-Plastic Bodies // Proc. of XXIX Summer School – Conference "Advanced Problems in Mechanics". St. Petersburg. 2002. P. 36–48.

2. Zhilin P.A. Mathematical theory of non-elastic media // Uspehi mechaniki (Advances in mechanics). Vol. 2. N 4. 2003. P. 3–36. (In Russian.)

3. Zhilin P.A. On the general theory of non-elastic media // Mechanics of materials and strength of constructions. Proc. of St. Petersburg State Polytechnical University. N 489. 2004. P. 8–27. (In Russian.)

Micro-polar theory for binary media

Micro-polar theory for binary media is developed (2003) [1, 2]. The media consists from liquid drops and fibers. The liquid is assumed to be viscous and non-polar, but with nonsymmetric stress tensor. The fibers are described by nonsymmetric tensors of force and couple stresses. The forces of viscous friction are taken into account. The second law of thermodynamics is formulated in the form of two inequalities, where the components of the binary media can have different temperatures.

1. Altenbach H., Naumenko K., Zhilin P. A micro-polar theory for binary media with application to phase-transitional flow of fiber suspensions // Continuum Mechanics and Thermodynamics. 2003. Vol. 15. N 6. P. 539–570.

2. Altenbach H., Naumenko K., Zhilin P.A. A micro-polar theory for binary media with application to flow of fiber suspensions // Proc. of XXX Summer School – Conference "Advanced Problems in Mechanics". St. Petersburg. 2003. P. 39–62.

Development of mathematical methods

The theory of symmetry for tensor quantities is developed. The new definition for tensor invariants is given (2005) [1, 2]. This definition coincides with the traditional one only for the Euclidean tensors. It is shown that any invariant can be obtained as a solution of a differential equation of the first order. The number of independent solutions of this equation determines the minimum number of invariants necessary to fix the system of tensors as a solid unit.

1. Zhilin P.A. Modified theory of symmetry for tensors and their invariants // Izvestiya VUZov. Severo-Kavkazskii region. Estestvennye nauki (Transactions of Universities. South of Russia. Natural sciences). Special issue “Nonlinear Problems of Continuum Mechanics”. 2003. P 176–195. (In Russian).

2. Zhilin P.A. Symmetries and Orthogonal Invariants in Oriented Space // Proc. of XXXII Summer School – Conference “Advanced Problems in Mechanics”. St. Petersburg. 2005. P. 470–483.

Piezoelasticity

Equations of piezoelasticity are obtained (2002–2005) [1, 2]. These equations contain as particular cases several theories, two among them are new. The proposed general theory is based on the model of micro-polar continuum. The main equations are derived from the fundamental laws of the Eulerian mechanics. These equations contain nonsymmetric tensors of stress and couple stress.

1. Kolpakov Ja. E., Zhilin P.A. Generalized continuum and linear theory of the piezoelectric materials // Proc. of XXIX Summer School – Conference “Advanced Problems in Mechanics”. St. Petersburg. 2002. P. 364–375.

2. Zhilin P.A., Kolpakov Ya.E. A micro-polar theory for piezoelectric materials // Lecture at XXXIII Summer School – Conference “Advanced Problems in Mechanics”. St. Petersburg, Russia. 2005. Current book. Vol. 2.

Ferromagnetism

The theory of the nonlinear elastic Kelvin medium whose particles perform translational and rotational motion, with large displacements and rotations, and may freely rotate about their axes of symmetry, is proposed. The constitutive and dynamic equations are obtained basing upon the fundamental laws of the Eulerian mechanics. The exact analogy is established between the equations for a particular case of Kelvin medium and the equations of elastic ferromagnetic insulators in the approximation of quasimagnetostatics (1998–2001) [1–3]. It is shown that the existing theories of magnetoelastic materials did not take into account one of the couplings between magnetic and elastic subsystem, which is allowed by fundamental principles. This coupling is important for the description of magnetoacoustic resonance, and may manifest itself in nonlinear theory as well as in the linear one for the case of anisotropic materials.

1. Grekova E.F., Zhilin P.A. Ferromagnets and Kelvin's Medium: Basic Equations and Magnetoacoustic Resonance // Proc. of the XXV–XXIV Summer Schools "Nonlinear Oscillations in Mechanical Systems". Vol. 1. St. Petersburg. 1998. P. 259–281.

2. Grekova E.F., Zhilin P.A. Equations for non-linear elastic polar media and analogies: Kelvin medium, non-classical shells, and ferromagnetic insulators // Izvestiya VUZov. Severo-Kavkazskii region. Estestvennye nauki (Transactions of Universities. South of Russia. Natural sciences). Special issue "Nonlinear Problems of Continuum Mechanics". 2000. P 24–46. (In Russian).

3. Grekova E.F., Zhilin P.A. Basic equations of Kelvin's medium and analogy with ferromagnets // Journal of elasticity. Vol. 64. (2001) P. 29–70.

Compilers: D.A. Indeitsev, E.A. Ivanova, A.M. Krivtsov.

Classical and Modified Electrodynamics*

Abstract

Analysis of classical Maxwell's equations reveals following peculiarities: 1. In a general case there exists no solution for the classical system; 2. If there is a solution, it can be represented as a superposition of transversal waves propagating with light's velocity "c" and quasi-electrostatic conditions setting in instantly over the whole space.

It means, that notwithstanding the settled opinion the Maxwell's equations are not compatible with the special theory of relativity. Modified Maxwell's equations are given in this paper possessing following features:

1. There exists always a solution to them;
2. This solution is a superposition of transversal and longitudinal waves, the latter propagating with a velocity $c_1 > c$;
3. Electrostatic conditions are setting in by passing of the longitudinal wave;
4. If the scalar potential is equal to zero, solutions for classical and modified systems coincide, i.e. both systems give the same description for magnetic fields;
5. In a general case solution for the modified system transforms itself into solution for the classical system by $c_1 \to \infty$.

Classical as well as modified systems are shown to be not suitable for a correct description of interactions between the nucleus and electrons of an atom. A way to creating a new electrodynamics based on more strict principles not using quantification is shown.

1 Classical electrodynamics

Classical Maxwell's equations are described and interpreted in this section using mechanical terms. It was exactly electrodynamics, which was the source of the opinion about

*Zhilin P.A. Classical and Modified Electrodynamics // Proc. of Int. Conf. New Ideas in Natural Sciences. Part I – Physics. P. 73–82. 1996. St. Petersburg. Russia. 642 p. (Zhilin P.A. Reality and Mechanics // Proceedings of XXIII Summer School "Nonlinear Oscillations in Mechanical Systems", St. Petersburg, Russia, 1996. P. 6–49. In Russian.)

mechanistical description of Universe being basically limited and useless for investigation of electromagnetic processes. Subsequently I intend to refute this point of view.

In the modern physics Maxwell's equations are considered to be something like divine revelation, thus being just postulated. In their canonical form they can be written as follows [3, p.76]

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\varepsilon_0 c}\mathbf{j}, \qquad (1)$$

where ρ represents density of charge, $\mathbf{j}$ being current's density, i.e. velocity of the charge's flow through a unit of area. The modern version is given here, which does not coincide with J.Maxwell's point of view: to Maxwell's opinion current is not necessarily connected with motion of charges. The latter circumstance will be shown to be quite significant. From (1) following condition of solvability can be obtained:

$$\nabla \cdot \mathbf{j} = -\partial\rho/\partial t. \qquad (2)$$

Remark. Physicists prefer to call equation (2) a law of charge conservation considering it to be a law of Nature. From mechanical point of view generally there exist no laws of conservation, but only balance equations for certain quantities. In particular, the local charge balance equation can be written as follows:

$$\nabla \cdot \mathbf{j} = -\partial\rho/\partial t + h, \qquad (3)$$

where h represents the volumetric speed density of the charge supplied to the given system. Even if there exist some conservation laws in Nature as a whole, they are absolutely useless for rational science, for we never examine Nature as a whole and never shall be able to do it. Mechanics and physics are investigating limited material systems being able to exchange everything, including charge, with their surroundings. Conservation laws exist for a very small class of isolated systems only. Therefore, it is in no way acceptable to consider equation (2) as a law of Nature — this is just a necessary condition of solvability for classical Maxwell's equations. It plays no such role for modified Maxwell's equations described in the next section. For them it is possible to use (3) instead of (2).

"In the course of time an opinion has formed itself on deduction of the Maxwell's equations being impossible on the basis of mechanical equations regardless of any generalizations made. Most theorists are convinced today: there is no need to deduct these equations, which are to be considered as a very successful, almost perfectly exact description of electromagnetic processes". This is a quotation from a quite old book being far from indisputable [4, pp.155-156]. Nevertheless, these words reflect quite correct the contemporary position. It suffices to take a cursory look at the system (1) to feel a doubt about its impeccability. First, there is a problem concerning treatment of the current. According to it, vector $\mathbf{j}$ is defined as speed of charge's flow through an unit of area. The system (1) is overdefined by that being insolvable in a general case. This conclusion follows just from the fact, that there are eight equations (ρ and $\mathbf{j}$ are given!) for six coordinates of the vectors $\mathbf{E}$ and $\mathbf{B}$. It is to be remarked though: the third equation of (1) follows from other three equations, if it is true for any moment of time. So in fact there are seven equations for six unknown quantities contained in (1). This contradiction can be eliminated by refusing the above treatment of the current $\mathbf{j}$. Consequences of such a refusal will be discussed in section 7. Main claims arising in connection with the system

(1) concern conclusions being obtained by mechanical interpretation of this system. Let us represent it in another, but equivalent form.

Now we introduce vector **u** satisfying following conditions:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{u}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{u}\,. \tag{4}$$

The second and the third equations of the system (1) allow introduction of such a vector. It is known, that every vector u can be represented as follows:

$$\mathbf{u} = \nabla\varphi + \nabla \times \mathbf{\Phi}\,, \qquad \nabla \cdot \mathbf{\Phi} = 0, \tag{5}$$

where potential φ is defined up to an arbitrary function of co-ordinates. It means, that addition to φ of an arbitrary function of co-ordinates does not change electric and magnetic field. By taking into account (4) and (5) it follows from the first equation of the system (1):

$$\Delta\varphi = q, \qquad \partial q/\partial t = -c\rho/\varepsilon_0, \tag{6}$$

where function q is defined up to an arbitrary function of co-ordinates. Thus, it remains to examine the fourth equation of (1). Let us represent the current **j** in following form:

$$\mathbf{j} = \nabla\varphi_* + \nabla \times \mathbf{\Phi}_*\,, \qquad \varphi_* = \frac{\varepsilon_0}{c}\frac{\partial^2\varphi}{\partial t^2}, \qquad \nabla \cdot \mathbf{\Phi}_* = 0. \tag{7}$$

Inserting these expressions into the last equation of (1), we obtain:

$$\Delta\mathbf{\Phi} - \frac{1}{c^2}\frac{\partial^2\mathbf{\Phi}}{\partial t^2} + \frac{1}{\varepsilon_0 c}\mathbf{\Phi}_* = 0. \tag{8}$$

It is easy to make certain, that the system (4)–(8) is exactly equivalent to the system (1). It allows a simple mechanical interpretation. Let us notify: according to (7) a current is not necessarily caused by motion of charges. However, in the last case the current can be treated as motion of charges too by considering electromagnetic field as consisting from two media, one of them being a continuum of negatively charged particles and the other — a continuum of positively charged particles, total density of charge being equal to zero. A current is nothing else as motion of one medium in respect to the other in this case. By such a treatment there exists no vacuum at all.

Let us now collect all the equations in one table containing two columns. In the left column there are equations of electrodynamics there, in the right one — equations of the linear dynamical theory of elasticity [5].

Electrodynamics	*Theory of elasticity*	
$\mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{u}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{u}\,, \qquad \mathbf{u} = \nabla\varphi + \nabla \times \mathbf{\Phi}\,, \qquad \nabla \cdot \mathbf{\Phi} = 0$		(I)
$\mathbf{j} = \nabla\varphi_* + \nabla \times \mathbf{\Phi}_*\,, \quad \nabla \cdot \mathbf{\Phi}_* = 0$ A.	$\dfrac{1}{\mu}\mathbf{F} = \nabla\tilde{\varphi} + \nabla \times \tilde{\mathbf{\Phi}}\,, \quad \nabla \cdot \tilde{\mathbf{\Phi}} = 0$ B.	(II)
$\Delta\mathbf{\Phi} - \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{\Phi}}{\partial t^2} = -\dfrac{1}{\varepsilon_0 c}\,\mathbf{\Phi}_*$ A.	$\Delta\mathbf{\Phi} = \dfrac{1}{c_2^2}\dfrac{\partial^2\mathbf{\Phi}}{\partial t^2} - \tilde{\mathbf{\Phi}}$ B.	(III)
$\Delta\varphi = q, \qquad \partial q/\partial t = -c\rho/\varepsilon_0\,,$ $\varphi_* = \dfrac{\varepsilon_0}{c}\dfrac{\partial^2\varphi}{\partial t^2}$ A.	$\Delta\varphi - \dfrac{1}{c_1^2}\dfrac{\partial^2\varphi}{\partial t^2} = -\left(\dfrac{c_2}{c_1}\right)^2\tilde{\varphi}$ B.	(IV)

In this table $c_1^2 = (\lambda + 2\mu)/\rho_*$, $c_2^2 = \mu/\rho_*$, ρ_* is mass density of the medium, λ and μ are Lame's constants, $\mathbf{F}$ representing volumetric force. Velocities c_1 and c_2 define the speeds of expansion and shear waves respectively. Positiveness of deformation energy requires fulfillment of the condition $c_1^2 > 4c_2^2/3$.

I would like to remember: in contrast to electrodynamical equations theorems of solution existence are proved under sufficiently generalized assumptions for equations of the elasticity theory. Let us now interpret the equations of electrodynamics. Vector $\mathbf{u}$ in the line I is representing potential for electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields. In the elasticity theory $\mathbf{u}$ is a vector of small displacements, $\mathbf{E}$ being normalized speed taken with reverse sign and $\mathbf{B}$ representing the rotor of the displacement vector being rarely used in theory of elasticity, but quite suitable for applying. Second line (II) requires no comments except of verification, that the current $\mathbf{j}$ in electrodynamics is analogous to the volumetric force in theory of elasticity. Analogy contained in line (III) will be obvious by assumptions:

$$c_2 = c\,, \qquad \tilde{\mathbf{\Phi}} \longleftrightarrow \frac{1}{\varepsilon_0 c}\,\mathbf{\Phi}_*\,.$$

Distinctions are most pronounced in the line (IV). Just the equations listed in this line define differences between electrodynamics and mechanics. In physics they are considered to be a proof of impossibility to interpret electrodynamics from mechanistical point of view, thus proving limited nature of mechanics. It would be more natural though, to admit some strangeness inherent in equations of electrodynamics and not in those of mechanics. In fact, the meaning of equation placed in the left column of the line (IV) is quite obvious. Potential φ exists for every quantity $\tilde{\varphi}$, i.e. for every volumetric force $\mathbf{F}$. Situation is different in electrodynamics. Current $\mathbf{j}$ cannot be defined arbitrarily, but is calculated (partially) from the potential φ, otherwise there may be no solution for an electrodynamical problem. This circumstance gives ground to doubts concerning "almost perfectly exact description of electromagnetic processes" with Maxwell's equations. However, considerations presented do not suffice. In contrast to the elasticity theory potential φ is not a solution of the wave equation in electrodynamics. This means electrodynamical potential φ be setting in at an instant over the whole space. In other words, Maxwell's equations lead to an infinitely high speed of signal's propagation, which contradicts scandalously to special theory of relativity (STR). Thus, STR and Maxwell's electrodynamics are not compatible with each other. There are obviously unremovable contradictions between STR and equations of the elasticity theory as well, the latter giving two values for speed of the signal's propagation. Generally, any theory giving more than one value for velocity of wave propagation cannot be compatible with the STR. To reveal more clearly analogies in equations (IV.A) and (IV.B) let us rewrite equations (IV.A) in an equivalent form:

$$\Delta\varphi - \frac{1}{c_1^2}\frac{\partial^2\varphi}{\partial t^2} = q - \frac{c}{\varepsilon_0 c_1^2}\,\varphi_*, \qquad \frac{c}{\varepsilon_0}\,\varphi_* = \frac{\partial^2\varphi}{\partial t^2}, \qquad \frac{\partial q}{\partial t} = -\frac{c}{\varepsilon_0}\,\rho\,. \tag{9}$$

First of these equations is quite analogous to equation (IV.B), provided

$$\left(\frac{c}{c_1}\right)^2 \tilde{\varphi} \;\longleftrightarrow\; \frac{c}{\varepsilon_0 c_1^2}\,\varphi_* - q\,.$$

Now it is easy to establish analogy between the volumetric force $\mathbf{F}$, the current and the charge:

$$\frac{1}{\mu}\mathbf{F} \;\longleftrightarrow\; \frac{1}{\varepsilon_0 c}\,\mathbf{j} - \left(\frac{c_1}{c}\right)^2 \nabla q$$

Assumption $\varphi_* = (\varepsilon_0/c)\,\partial^2\varphi/\partial t^2$ means a compulsory definition of a part of the volumetric force. Such an assumption appears not too convincing in mechanics as well as in electrodynamics. Nevertheless, mechanistic interpretation of the classical electrodynamics equations is obvious already, and there is no need to discuss the matter any more. Situation becomes entirely simple, if there are no charges and currents, or no volumetric forces in the elasticity theory. In this case line (IV) can be written as follows:

$$\Delta\varphi = 0 \quad \text{(IV.A)}, \qquad \Delta\varphi = \frac{1}{c_1^2}\frac{\partial^2\varphi}{\partial t^2} \quad \text{(IV.B)}$$

Equation (IV.B) transforms itself into (IV.A) by $c_1 \to \infty$. In this case the Maxwell's equations become identical to those describing oscillations of an incompressible medium, which was noted by Maxwell himself [1, p.784].

Concluding this section we want to underline: mechanical analogies for Maxwell's equations have proved themselves to be simple enough and well known to all mechanicians.

2 Modified Maxwell's equations

As noted above, classical Maxwell's equations have a grave drawback: they lead to an infinite high velocity of signal's propagation. Unfortunately, this is not the only defect of classical equations and even not the most important one, as will be shown later. Here we shall adduce a modified system of Maxwell's equations providing only finite velocities for propagation of any signals. By refusing connection described by the second equation of (9) we obtain following system:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{u}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{u}, \qquad \mathbf{u} = \nabla\varphi + \nabla \times \mathbf{\Phi}, \qquad \nabla \cdot \mathbf{\Phi} = 0; \tag{10}$$

$$\Delta \mathbf{\Phi} - \frac{1}{c^2}\frac{\partial^2 \mathbf{\Phi}}{\partial t^2} = -\frac{1}{\varepsilon_0 c}\mathbf{\Phi}_*; \tag{11}$$

$$\Delta\varphi - \frac{1}{c_1^2}\frac{\partial^2\varphi}{\partial t^2} = q - \frac{c}{\varepsilon_0 c_1^2}\varphi_*, \qquad \frac{\partial q}{\partial t} = -\frac{c}{\varepsilon_0}\rho, \qquad c_1^2 > 4c^2/3, \tag{12}$$

where the current is expressed by the formula:

$$\mathbf{j} = \nabla\varphi_* + \nabla \times \mathbf{\Phi}_*, \qquad \nabla \cdot \mathbf{\Phi}_* = 0. \tag{13}$$

The system (10)–(13) can be rewritten in a form more convenient for electrodynamics:

$$\nabla \cdot \mathbf{E} = \frac{\rho_*}{\varepsilon_0}, \quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\varepsilon_0 c}\mathbf{j}_*, \tag{14}$$

where

$$\rho_* = \rho + \frac{1}{c_1^2}\frac{\partial}{\partial t}\left(\varphi_* - \frac{\varepsilon_0}{c}\frac{\partial^2\varphi}{\partial t^2}\right), \qquad \mathbf{j}_* = \mathbf{j} - \nabla\left(\varphi_* - \frac{\varepsilon_0}{c}\frac{\partial^2\varphi}{\partial t^2}\right). \tag{15}$$

It is necessary to add equations (12) and (13) to these relations to obtain a closed system.

System (14) appears to be like the system (1), but the meaning of it is significantly different. This difference is especially noticeable for areas, where ρ and $\mathbf{j}$ are equal to zero.

$$\rho = 0, \quad \mathbf{j} = \mathbf{0} \quad \Rightarrow \quad \varphi_* = c(t), \quad \mathbf{\Phi}_* = \mathbf{0}.$$

According to classical system (1) we shall have for this case: $\nabla \cdot \mathbf{E} = 0$. This is exactly the relation infinite velocity of signal's propagation is hidden in. Let us imagine following situation. Suppose, there existed two point charges by $t < 0$, having equal amount, but different signs and being situated at the same point by $t \leq 0$. In that case $\mathbf{E} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$. for $t \leq 0$. At the moment $t = 0$ these charges begin to scatter. It is easy to make certain, that potential φ will have to differ from zero by $t > 0$. By representing fields $\mathbf{E}$ and $\mathbf{B}$ with waves there would exist an area located far away from charges, where fields $\mathbf{E}$ and $\mathbf{B}$ did not come into existence yet. This area is separated from regions with existing fields $\mathbf{E}$ and $\mathbf{B}$ with a certain movable surface Σ being called wave front (see Figure 1). Let us choose a closed area bordered by the surface S. According to classical equations we shall have inside of this surface:

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0.$$

These conditions are true everywhere for transversal waves, including interior of the surface S, i.e. they are true for $\mathbf{B}$ and a part of $\mathbf{E}$ represented by a transversal wave. But potential φ cannot be represented with a transversal wave, therefore it is impossible

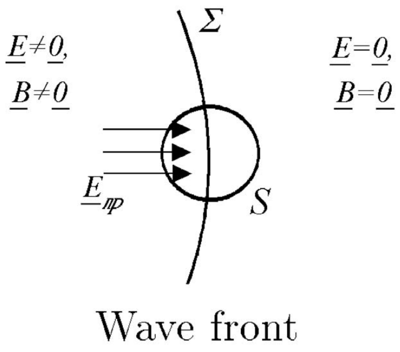

Figure 1: Wave front

for the quantity $\nabla \cdot \mathbf{E}$ to be equal to zero at the wave front, because there is something coming inside of S, but nothing comes out of it. The contradiction disappears, if we agree, that φ is not a wave and it does not have a wave front. This assumption conforms to classical electrodynamics — potential φ is setting in instantaneously over the whole space. According to modified system $\nabla \cdot \mathbf{E} \neq 0$ even if there are no charges and currents present. In the next section we shall present examples showing clearly all the points mentioned before.

Modified system (10)–(13) cannot be worse than the classical one, for the latter is contained in the first as a special case. A most "strange" feature of the system (10)–(13) is presence of waves propagating with a velocity $c_1 > c$. In the next section it will be shown, that these waves are responsible for setting in of electrostatic fields. Therefore, the modified system eliminates the abyss between electrostatics and electrodynamics inherent in classical Maxwell's equations. These equations do not allow it to infer electrostatics from a dynamical problem, electrostatics being quite a system "closed in itself". The system (10)–(13) can be considered mathematically irreproachable. How real are the waves described by equation (12)? What is the value of the velocity c_1? There are no answers to these questions yet. Pure intuition confirms existence of transversal waves (12). For myself, I doubt it in no way, for there arise unsolvable problems otherwise. Existence of waves propagating faster than light is inquestionable from experimental point of view. This fact was first established by N.A.Kosyrev [2] and then confirmed with all possible thoroughness by Academician M.M.Lavrentyev and his colleagues [6,7]. The essence of Kosyrev's experiment consists in following. He has developed a sensor to detect radiation of different types without subtilizing the nature of it. Using this sensor, Kosyrev has fixed radiation flows coming from stars. By directing the telescope at a visible star he would fix a local maximum of radiation intensity. But then he has made the most staggering discovery: he would fix a more intensive radiation by directing the telescope at the point on the sky, where the star would be really positioned at the moment of observation. Of course, there would be no star to be seen at that point, because the light coming from it will reach the Earth in distant future only. One can agree or disagree with explanations given for that by N.A.Kosyrev. But the fact of existence of radiation propagating much faster than light appears incontestable. Sure, there are no definite reasons to assume equation (12) be describing exactly that radiation, but one cannot exclude such a possibility as a matter of principle. In any case, special experiments are needed to verify the system (10)–(13) and to define the velocity c_1. It is important to note: any experimental data being explainable by classical equations would be full explained by modified equations as well.

So, the modified system cannot be worse than the classical one. Moreover, it is much better theoretically. Nevertheless, fundamental completeness of the classical as well as of the modified system appears more than doubtful. It is clear intuitively, that magnetic phenomena are being described by these systems incompletely and in a heavy distorted form, if at all. I cannot go into details here and shall confine myself to obvious remarks demonstrating fundamental incompleteness of the Maxwell's equations. To this end it is necessary to take into account facts firmly established by experimental physics.

Fact one. Interactions between the nucleus and the electrons of an atom must be of electromagnetic nature, thus they are to be described with equations of electrodynamics.

Fact two. Any atom possesses a mixed discrete-continuous spectrum to be defined experimentally.

Striving for explanation of these facts has led to establishment of quantum physics. From the point of view adopted in this paper integrity of an atom and its structure (but not the structure of the nucleus or electrons) is to be explained using equations for the second ether, i.e. equations of electrodynamics, but sure not classical electrodynamics. It is known in mechanics (see for example [8]), that mixed spectra appear by investigation of some specific problems provided presence of two main factors. First of them: presence of a boundless medium described by an operator with continuous spectrum disposed above a certain frequency (cut-off frequency). Second ether plays the role of such a boundless medium. Equations describing oscillations of an infinite beam (or a string) on an elastic bedding can be cited as an example of equations being defined on a boundless medium and having a cut-off frequency. To get a cut-off frequency from an electrodynamical equation for a boundless medium it is necessary to take into account spinorial motions being responsible for magnetic phenomena. The second factor: discrete spectrum appears below the cut-off frequency, if there are discrete particles inserted into the field of operator with continuous spectrum. Nucleus and electrons play the role of such particles. By inserting nucleus and electrons into classical or modified electromagnetic field there will appear no discrete (separated) frequencies, because the system (1) as well as the system (10)–(13) do not have any cut-off frequencies. The latter would appear in waveguides, but that is not a boundless medium anymore. Thus, electrodynamical equations are to be significantly changed to explain the structure of an atom. Exactly this is being done in quantum electrodynamics, but there are other ways remaining in the framework of the classical mechanics.

3 Electromagnetic field of a growing point charge

Some facts inherent in classical electrodynamics give rise to considerable doubts by everyone educated on traditions of classical mechanics. First of all, it is true for electrostatics, which is included in electrodynamics as a thing for itself. Every static problem in mechanics can be derived from an appropriate dynamical problem by transition to a limit. Static conditions are setting in over a body by means of certain waves. That is not so in electrodynamics: electrostatic field sets in instantaneously over the whole space. There is another fact. R.Feinmann writes [3, p.78]: "Laws of physics do not answer the question: "What will happen by a sudden appearance of a charge at a given point? Which electromagnetic effects will be observed?" There can be no answer to that, because our equations deny the very possibility of such events. If it would happen, we would need new laws, but we cannot say, what they would be like to...". It sounds very strange for a mechanician. In mechanics we suddenly apply forces of unknown sources and observe the system's reaction to these forces. Moreover, the main equations have to be solvable by arbitrarily determined external forces irrespective of the very possibility for such forces to exist. Charges and currents in electrodynamics are analogous to volumetric forces in the elasticity theory. Therefore, from mechanical point of view a satisfactory electrodynamical theory is just obliged to give a simple answer to Feinmann's question.

Suppose, there is a charge coming into existence at a given point (at the initial point of co-ordinate system). We assume this charge called a point source to be changing according to following law:

$$Q(t) = Q_0[1 - e(t)]\,, \qquad e(t) \equiv \exp\left(-2\pi\frac{t}{\tau}\right) \qquad t \geq 0$$

It is required to define disturbances of electromagnetic field connected with this source. Physicists prefer to call these disturbances just electromagnetic field as such. The problem formulated was investigated by R.Feinmann for an arbitrary function Q(t) [3, pp.145-147]. The reader can compare solution represented below with that of R.Feinmann.

The problem possesses spherical symmetry, i.e. there exist two planes of the mirror symmetry. Therefore, all quantities being represented by axial vectors, must be equal to the zero vector:

$$\mathbf{B} = \mathbf{0}\,, \qquad \mathbf{\Phi} = \mathbf{0}\,, \qquad \mathbf{\Phi}_* = \mathbf{0}\,.$$

Firstly, let us try to solve this problem using the classical system (1) under assumption of current being a motion of charges. As there are no moving charges there, $\mathbf{j} = \mathbf{0}$. We shall construct the solution using spherical co-ordinate system. In this case $\mathbf{E} = \mathcal{E}(r,t)\,\mathbf{e}_r$. Divergence $\mathbf{E}$ is equal to zero by $r \neq 0$, therefore

$$\nabla \cdot \mathbf{E} = \frac{\partial \mathcal{E}}{\partial r} + \frac{2}{r}\,\mathcal{E} = 0 \quad \Rightarrow \quad \mathcal{E}(r,t) = \frac{C(t)}{r^2}\,.$$

Using theorem of Gauss, we can define $C(t)$ and then the field $\mathbf{E}$:

$$\mathbf{E} = \frac{Q(t)}{4\pi\varepsilon_0 r^2}\,\mathbf{e}_r\,. \tag{16}$$

From the last equation of (1) we conclude, that $\partial\mathbf{E}/\partial t = \mathbf{0}$. Hence, there is no solution of the classical system (1) by $\mathbf{j} = \mathbf{0}$, because $dQ/dt \neq 0$. This is just the case, which was investigated by R.Feinmann, thus, the formula (21.13) cannot be considered to be a solution.

If we adopt a quite forced assertion about current not being necessarily connected with motion of charges and define the current as an additional unknown quantity, we shall be able to solve the classical system, because we obtain from the last equation of (1) and relation (16) for that case:

$$\mathbf{j} = -\frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{4\pi r^2}\,\frac{dQ}{dt}\,\mathbf{e}_r\,.$$

Despite of existence of a formal solution it cannot be considered physically satifactory, for it sets in instantaneously over the whole space.

Let us now investigate the problem using the modified system (10)–(13). Here the current is assumed to be motion of charges, i.e. the quantity $\mathbf{j}$ is determined. We can write for that case: $\mathbf{j} = \mathbf{0} \;\Rightarrow\; \varphi_* = 0,\;\; \mathbf{\Phi}_* = \mathbf{0}\,.$

Potential $\varphi = \varphi(r,t)$ is described by equation (12)

$$\Delta\varphi = \frac{1}{c_1^2}\,\frac{\partial^2\varphi}{\partial t^2} + q\,; \qquad \frac{\partial q}{\partial t} = -\frac{c}{\varepsilon_0}\,\rho\,.$$

Let us rewrite this equation for the function $\psi(z,t) = \partial\varphi/c\,\partial t$

$$\Delta\psi = \frac{1}{c_1^2}\,\frac{\partial^2\psi}{\partial t^2} - \frac{1}{\varepsilon_0}\,\rho\,. \tag{17}$$

We can define electric field using following formula:

$$\mathbf{E} = -\frac{1}{c}\,\frac{\partial}{\partial t}\nabla\varphi = -\nabla\psi = -\frac{\partial\psi}{\partial r}\,\mathbf{e}_r\,.$$

Classical theorem of Gauss is no more true for this case. Suppose, the initial point of the co-ordinate system is surrounded with a small spherical volume $V_r\,,\; r \to 0\,.$ By multiplying both sides of (17) with dV_r and integrating them over volume V_r we obtain:

$$\int_{V_r} \Delta\psi\, dV_r = \frac{1}{c_1^2}\,\frac{\partial^2}{\partial t^2}\int_{V_r} \psi\, dV_r - \frac{1}{\varepsilon_0}\,Q_0\,[1 - e(t)]\,. \tag{18}$$

Using the divergence theorem, we shall have:

$$\int_{V_r} \Delta\psi\, dV_r = \int_{S_r} \mathbf{e}_r\cdot\nabla\psi\, dS_r = -\int_{S_r} \mathbf{e}_r\cdot\mathbf{E}\, dS_r\,.$$

Assuming $r \to 0$, we can write:

$$\lim_{r\to 0}\int_{S_r} \mathbf{e}_r\cdot\mathbf{E}\, dS_r = \frac{Q_0}{\varepsilon_0}\,[1 - e(t)]\,. \tag{19}$$

This relation will replace the theorem of Gauss for us.

Let us write equation (17) for an area with $r \neq 0$

$$\frac{\partial^2 r\psi}{\partial r^2} = \frac{1}{c_1^2}\,\frac{\partial^2 r\psi}{\partial t^2} \quad\Rightarrow\quad r\psi(r,t) = f(r - c_1 t)\,,$$

Here it is taken into account, that no radiation is coming from infinity. As there was no field by $t = 0$, $f(s) = 0$ by $s \geq 0$. Consequently, the function $f(r - c_1 t)$ is different from zero only by negative values of the argument $s = r - c_1 t$, i.e. in the area $r < c_1 t$. Thus, we obtain a wave representation for the field $\mathbf{E}$:

$$\mathbf{E} = -\frac{\partial}{\partial r}\left[\frac{f(r - c_1 t)}{r}\right]\mathbf{e}_r = \left[\frac{f(r - c_1 t)}{r^2} - \frac{f'(r - c_1 t)}{r}\right]\mathbf{e}_r \,.$$

Now we can write:

$$\int_{S_r} \mathbf{e}_r \cdot \mathbf{E}\, dS_r = 4\pi\left[f(r - c_1 t) - r f'(r - c_1 t)\right].$$

By substitution this expression into (19) we get:

$$f(-c_1 t) = \frac{Q_0}{4\pi\varepsilon_0}\left[1 - e(t)\right].$$

Using this relation, we can define function f by negative values of the argument.

Finally we obtain following solution:

$$\mathbf{E}(r,t) = -\frac{Q_0}{4\pi\varepsilon_0}\,\mathbf{e}_r\,\frac{\partial}{\partial r}\begin{cases} \dfrac{1}{r}\left[1 - \exp\left(2\pi\,\dfrac{r - c_1 t}{c_1 \tau}\right)\right], & r \le c_1 t\,; \\ 0\,, & r \ge c_1 t\,. \end{cases} \qquad (20)$$

From this expression it can be easily seen, that a quasi-static solution (16) is setting in for the area $r < c_1(t - \tau)$ by $t > \tau$. This solution is given by the classical system (1) assuming presence of non-zero current. Current is absent in the solution (20).

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A General Model of Rigid Body Oscillator*

Abstract

The present discourse develops a new model named by a rigid body oscillator. In Eulerian mechanics this model plays the same role as the model of nonlinear oscillator in Newtonian mechanics. The importance of the introduction of the rigid body oscillator, i.e. a rigid body oscillator on an elastic foundation of general kind, into consideration was pointed out by many scientists. However the problem is not formalized up to now. In the paper all necessary for a mathematical description concepts are introduced. Some of them are new. The equations of motion are represented in unusual for dynamics of rigid body form, which has a clearly expressed simple structure but contain the nonlinearity of a complex kind. These equations give the very interesting object for the theory of nonlinear oscillations. The solutions of some problems are given. For the simplest case the exact solution was found by an essentially new method of an integration of basic equations.

1 Introduction

The nonlinear (linear) oscillator is the most important model of classical physics. An investigation of many physical phenomenons and a development of many methods of nonlinear mechanics had arisen in the science due to this model. At the same time it was recognized the necessity of construction of models with new properties. Especially it was important in quantum mechanics, where many authors pointed out that a new model must be something like a rigid body on an elastic foundation. However, such model was not created up to now. Why? The full answer on this question will be found by historians later.

A rigid body on an elastic foundation will be called the rigid body oscillator in what follows. A general model of such object can be used in many cases, for example, in mechanics of continuum multipolar media. For the construction of model the three new elements are needed: the vector of turn, the integrating tensor, and the potential torque. Let us briefly discuss these concepts.

An unusual situation takes place with the **vector of turn**. From the one side, the well-known theorem of Euler proves that any turn of the body can be realized as the

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turn around some unit vector $\mathbf{m}$ by a certain angle θ. Thus the turn can be described by the vector $\boldsymbol{\theta} = \theta\mathbf{m}$. This fact can be found in many books on mechanics. From the other side the same books claim that the vector $\theta\mathbf{m}$ is not a vector, and a description of a turn in terms of vector is impossible. May be by this reason a vector of turn has no applications in conventional dynamics of rigid body. However namely the vector of turn plays the main role in dynamics of rigid body on an elastic foundation.

Integrating tensor. In classical mechanics the linear differential form $\mathbf{v}dt$ is the total differential of the vector of position $\mathbf{v}dt = d\mathbf{R}$. It is not true for spinor movements. If the vector $\boldsymbol{\omega}$ is a vector of angular velocity, then the linear differential form $\boldsymbol{\omega}dt$ is not a total differential of the vector of turn. However, it can be proved that there exists a tensor $\mathbf{Z}$ that transforms the linear differential form $\boldsymbol{\omega}dt$ into the total differential $d\boldsymbol{\theta}$ of the vector of turn $\boldsymbol{\theta}$. This fact was established in the work [2]. The integrating tensor $\mathbf{Z}$ plays the decisive role for an introduction of **a potential torque**. The latter expresses an action of the elastic foundation on the rigid body. Thus it is an essential element of a general model of rigid body oscillator.

The basic equations of dynamics of rigid body oscillator contain a strong nonlinearity but their form is rather simple. These equations give the very interesting object for methods of nonlinear mechanics. In the paper some simple examples are considering. In particular a new method of integration of the basic equations is given in the case of simplest model.

Author hopes that the clarity of the mathematical formulas in the paper will be able to compensate a helplessness of its language of words.

2 Mathematical preliminaries

In the section certain aspects of the tensor of turn and the vector of turn will be briefly presented. Some initial definitions can be found in the paper [1].

2.1 Vector of turn

A vector of turn is the very old concept. It is difficult to find another concept, for which there exist so many inconsistent propositions as for the vector of turn. The latter plays the main role in the present work. Because of this it seems to be necessary to give the strict introduction of the vector of turn and to describe its basic properties. The introduction of the vector of turn is determined by the well-known statement of Euler: any turn can be represented as the turn around some axis $\mathbf{n}$ by the certain angle θ. The vector $\theta\mathbf{n}$, $|\mathbf{n}|=1$, is called the vector of turn. Note that two different mathematical concepts correspond to one physical (or geometrical) idea of turn. One of them is described by a tensor of turn and another is described by a vector of turn. Of course both of them are connected by a unique manner. For the turn-tensor we shall use the notation [1]

$$\mathbf{Q}(\theta\mathbf{n}) = (1-\cos\theta)\,\mathbf{n}\otimes\mathbf{n} + \cos\theta\mathbf{E} + \sin\theta\mathbf{n}\times\mathbf{E}. \quad (1)$$

An action of the tensor $\mathbf{Q}(\theta\mathbf{n})$ on the vector $\mathbf{a}$ can be expressed in the form

$$\mathbf{a}' = \mathbf{Q}(\theta\mathbf{n})\cdot\mathbf{a} = (\mathbf{a}\cdot\mathbf{n})\,\mathbf{n} + \cos\theta\,(\mathbf{a}-\mathbf{a}\cdot\mathbf{n}\mathbf{n}) + \sin\theta\mathbf{n}\times\mathbf{a}. \quad (2)$$

If $\mathbf{n} \times \mathbf{a} = \mathbf{0}$, then $\mathbf{a}' = \mathbf{a}$. If $\mathbf{a} \cdot \mathbf{n} = 0$, then we have

$$\mathbf{a}' = \cos\theta \mathbf{a} + \sin\theta \mathbf{n} \times \mathbf{a}.$$

This means that vector $\mathbf{a}'$ is the vector $\mathbf{a}$ turned around the vector $\mathbf{n}$ by the angle θ.

Representation (1) can be rewritten in another form

$$\mathbf{Q}(\boldsymbol{\theta}) = \mathbf{E} + \frac{\sin\theta}{\theta}\mathbf{R} + \frac{1-\cos\theta}{\theta^2}\mathbf{R}^2 = \exp\mathbf{R}, \tag{3}$$

where

$$\mathbf{R} = \boldsymbol{\theta} \times \mathbf{E}, \qquad \theta = |\boldsymbol{\theta}|. \tag{4}$$

The vector $\boldsymbol{\theta}$ in (3), (4) is called the vector of turn. Note that there exists a little difference between representations (1) and (3). In (1) the quantity θ is the angle of turn and can be both positive and negative. In (3) the quantity θ is the modulus of the vector of turn, i.e. the modulus of the angle of turn. such interpretation is possible since, for example, $\sin\theta/\theta = \sin|\theta| / |\theta|$. As a rule, representation (3) is more convenient for applications then expression (1). Let us consider a superposition of two turns

$$\mathbf{Q}(\boldsymbol{\theta}) = \mathbf{Q}(\boldsymbol{\varphi}) \cdot \mathbf{Q}(\boldsymbol{\psi}). \tag{5}$$

The vector of total turn $\boldsymbol{\theta}$ is connected with the vectors of turn $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ by the formulas

$$1 + 2\cos\theta = \cos\varphi + \cos\psi + \cos\varphi\cos\psi -$$
$$-2\frac{\sin\varphi}{\varphi}\frac{\sin\psi}{\psi}\boldsymbol{\varphi}\cdot\boldsymbol{\psi} + \frac{(1-\cos\varphi)}{\varphi^2}\frac{(1-\cos\psi)}{\psi^2}(\boldsymbol{\varphi}\cdot\boldsymbol{\psi})^2, \tag{6}$$

$$\begin{aligned} 2\frac{\sin\theta}{\theta}\boldsymbol{\theta} = &\left[\frac{\sin\varphi}{\varphi}(1+\cos\psi) - \frac{(1-\cos\varphi)}{\varphi^2}\frac{\sin\psi}{\psi}\boldsymbol{\varphi}\cdot\boldsymbol{\psi}\right]\boldsymbol{\varphi} + \\ &+\left[\frac{\sin\psi}{\psi}(1+\cos\varphi) - \frac{(1-\cos\psi)}{\psi^2}\frac{\sin\varphi}{\varphi}\boldsymbol{\varphi}\cdot\boldsymbol{\psi}\right]\boldsymbol{\psi} + \\ &+\left[\frac{\sin\varphi\sin\psi}{\varphi\psi} - \frac{(1-\cos\varphi)}{\varphi^2}\frac{(1-\cos\psi)}{\psi^2}\boldsymbol{\varphi}\cdot\boldsymbol{\psi}\right]\boldsymbol{\varphi}\times\boldsymbol{\psi}. \end{aligned} \tag{7}$$

Note that from expressions (3), (4) it follows

$$\mathbf{R} \cdot \boldsymbol{\theta} = \mathbf{0}, \qquad \mathbf{Q}(\boldsymbol{\theta}) \cdot \boldsymbol{\theta} = \boldsymbol{\theta}. \tag{8}$$

2.2 Integrating tensor

The vector of turn $\boldsymbol{\theta}(t)$ plays for spinor movements the same role as the vector of position $\mathbf{R}(t)$ for translation movements. In the latter case the translation velocity $\mathbf{v}$ can be found by means of simplest formula $\mathbf{v} = \dot{\mathbf{R}}(t)$. This means that the linear form $\mathbf{v}dt$ is the total differential of the vector of position. For spinor movements the situation is more complicated, since the linear form $\boldsymbol{\omega}dt$, where $\boldsymbol{\omega}$ is the vector of angular velocity, is not the total differential of the vector of turn $\boldsymbol{\theta}$. Thus it is necessary to find an integrating factor that transforms the linear form $\boldsymbol{\omega} dt$ into the total differential of vector of turn $d\boldsymbol{\theta}$. For this end let us consider the left Poisson equation [1]

$$\dot{\mathbf{Q}}(\boldsymbol{\theta}) = \boldsymbol{\omega} \times \mathbf{Q}(\boldsymbol{\theta}), \qquad \dot{f} \equiv df/dt. \tag{9}$$

This equation for the tensor of turn $\mathbf{Q}(\boldsymbol{\theta})$ is equivalent to a system of nine scalar equations but only three of them are independent. In order to find these independent equations it is possible to substitute expression (3) into equation (9). After some transformations the next equation can be derived

$$\dot{\boldsymbol{\theta}}(t) = \mathbf{Z}(\boldsymbol{\theta}) \cdot \boldsymbol{\omega}(t), \tag{10}$$

where

$$\mathbf{Z}(\boldsymbol{\theta}) = \mathbf{E} - \frac{1}{2}\mathbf{R} + \frac{1-g}{\theta^2}\mathbf{R}^2, \qquad g = \frac{\theta \sin\theta}{2(1-\cos\theta)}. \tag{11}$$

The tensor $\mathbf{Z}(\boldsymbol{\theta})$ will be called the integrating tensor in what follows. The nonsingular tensor $\mathbf{Z}$ has the determinant

$$\det \mathbf{Z}(\boldsymbol{\theta}) = \theta^2/2(1-\cos\theta) \neq 0.$$

The integrating tensor has a number useful properties. Let us describe some of them. First of all, the tensor $\mathbf{Z}(\boldsymbol{\theta})$ is an isotropic function of the vector of turn $\boldsymbol{\theta}$. This means that

$$\mathbf{Z}(\mathbf{S} \cdot \boldsymbol{\theta}) = \mathbf{S} \cdot \mathbf{Z}(\boldsymbol{\theta}) \cdot \mathbf{S}^T, \qquad \forall \mathbf{S} : \mathbf{S} \cdot \mathbf{S}^T = \mathbf{E}, \qquad \det \mathbf{S} = 1. \tag{12}$$

If $\mathbf{S} = \mathbf{Q}(\boldsymbol{\theta})$, then from (12) and (8) it follows

$$\mathbf{Z}(\boldsymbol{\theta}) \cdot \mathbf{Q}(\boldsymbol{\theta}) = \mathbf{Q}(\boldsymbol{\theta}) \cdot \mathbf{Z}(\boldsymbol{\theta}).$$

Besides, it can be checked the identity

$$\mathbf{Z}^T(\boldsymbol{\theta}) = \mathbf{Q}(\boldsymbol{\theta}) \cdot \mathbf{Z}(\boldsymbol{\theta}) = \mathbf{Z}(\boldsymbol{\theta}) \cdot \mathbf{Q}(\boldsymbol{\theta}). \tag{13}$$

For the right angular velocity $\boldsymbol{\Omega} = \mathbf{Q}^T(\boldsymbol{\theta}) \cdot \boldsymbol{\omega}$ — see [1] — from expressions (10) and (13) it follows

$$\dot{\boldsymbol{\theta}}(t) = \mathbf{Z}^T(\boldsymbol{\theta}) \cdot \boldsymbol{\Omega}(t). \tag{14}$$

This equation is equivalent to the right Poisson equation [1]. In the explicit form equations (10) and (14) can be rewritten by such manner

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega}), \qquad \boldsymbol{\theta}\,|_{t=0} = \boldsymbol{\theta}_0, \tag{15}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\Omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\Omega}), \qquad \boldsymbol{\theta}\,|_{t=0} = \boldsymbol{\theta}_0. \tag{16}$$

Problem (15) is the left Darboux problem[1]. If the left angular velocity is known, then the vector of turn (and therefore the turn-tensor) can be found as the solution of problem (15). It is much more simple task (at least for numerical analysis) then a solution of the conventional Riccati equation. The same can be said with respect to the right Darboux problem (16). Expressions (15) and (16) can be rewritten in the equivalent form

$$\dot{\theta} = g\boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1-g}{\theta}\dot{\theta}\boldsymbol{\theta}, \tag{17}$$

$$\dot{\theta} = g\boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\Omega} + \frac{1-g}{\theta}\dot{\theta}\boldsymbol{\theta}. \tag{18}$$

Here we take into account the identity

$$\boldsymbol{\theta} \cdot \boldsymbol{\omega} = \boldsymbol{\theta} \cdot \boldsymbol{\Omega} = \boldsymbol{\theta} \cdot \dot{\boldsymbol{\theta}} = \theta\dot{\theta}$$

Sometimes it is more convenient to use an inverse form of equations (10) and (14)

$$\boldsymbol{\omega}(t) = \mathbf{Z}^{-1}(\boldsymbol{\theta}) \cdot \dot{\boldsymbol{\theta}}(t), \qquad \boldsymbol{\Omega}(t) = \mathbf{Z}^{-T}(\boldsymbol{\theta}) \cdot \dot{\boldsymbol{\theta}}(t), \tag{19}$$

where

$$\mathbf{Z}^{-1}(\boldsymbol{\theta}) = \mathbf{E} + \frac{1-\cos\theta}{\theta^2}\mathbf{R} + \frac{\theta - \sin\theta}{\theta^3}\mathbf{R}^2. \tag{20}$$

2.3 Potential torque

Let us introduce a concept of potential torque. This concept is necessary for a statement and an analysis of many problems. Nevertheless a general definition of potential torque is absent in literature.

Definition: Torque $\mathbf{M}(t)$ is called potential if there exists scalar function $U(\boldsymbol{\theta})$ depending on a vector of turn such that the next equality is valid

$$\mathbf{M} \cdot \boldsymbol{\omega} = -\dot{U}(\boldsymbol{\theta}) = -\frac{dU}{d\boldsymbol{\theta}} \cdot \dot{\boldsymbol{\theta}}. \tag{21}$$

Making use of equation (10) this equality can be rewritten in the form

$$\left(\mathbf{M} + \frac{dU}{d\boldsymbol{\theta}} \cdot \mathbf{Z}\right) \cdot \boldsymbol{\omega} = \mathbf{0}.$$

This equality must be satisfied for any vector $\boldsymbol{\omega}$. It is possible if and only if

$$\mathbf{M} = -\mathbf{Z}^{T}(\boldsymbol{\theta}) \cdot \frac{dU}{d\boldsymbol{\theta}} + \mathbf{f}(\boldsymbol{\theta}, \boldsymbol{\omega}) \times \boldsymbol{\omega}, \tag{22}$$

where $\mathbf{f}(\boldsymbol{\theta}, \boldsymbol{\omega})$ is some functional of vectors $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$.

Definition: a torque $\mathbf{M}$ is called positional if $\mathbf{M}$ depends on the vector of turn $\boldsymbol{\theta}$ only. For the positional torque $\mathbf{M}(\boldsymbol{\theta})$ we have

$$\mathbf{M}(\boldsymbol{\theta}) = -\mathbf{Z}^{T}(\boldsymbol{\theta}) \cdot \frac{dU(\boldsymbol{\theta})}{d\boldsymbol{\theta}}. \tag{23}$$

Let us show two simple examples.

If the potential function has a form of an isotropic function of a vector of turn

$$U(\boldsymbol{\theta}) = F\left(\theta^2\right),$$

then from expression (23) it follows

$$\mathbf{M}(\boldsymbol{\theta}) = -2\frac{dF(\theta^2)}{d(\theta^2)}\boldsymbol{\theta}.$$

Let the potential function has the simplest form

$$U(\boldsymbol{\theta}) = C\mathbf{k}\cdot\boldsymbol{\theta}, \qquad C = const, \qquad \mathbf{k} = const.$$

However, for the torque we have rather complex expression

$$\mathbf{M} = -C\mathbf{Z}^T\cdot\mathbf{k} = -C\left[\mathbf{k} + \frac{1}{2}\boldsymbol{\theta}\times\mathbf{k} + \frac{1-g}{\theta^2}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\mathbf{k})\right].$$

Let there be given a unit vector **k**.

Definition: the potential $U(\boldsymbol{\theta})$ is called transversally isotropic with the axis of symmetry **k** if the equality

$$U(\boldsymbol{\theta}) = U[\mathbf{Q}(\alpha\mathbf{k})\cdot\boldsymbol{\theta}]$$

holds good for any tensor of turn $\mathbf{Q}(\alpha\mathbf{k})$.

It can be proved that a general form of a transversally isotropic potential can be expressed as a function of two arguments

$$U(\boldsymbol{\theta}) = F\left(\mathbf{k}\cdot\boldsymbol{\theta}, \theta^2\right). \tag{24}$$

For this potential one can derive the expression

$$\mathbf{M}(\boldsymbol{\theta}) = -2\frac{\partial F}{\partial(\theta^2)}\boldsymbol{\theta} - \frac{\partial F}{\partial(\mathbf{k}\cdot\boldsymbol{\theta})}\mathbf{Z}^T\cdot\mathbf{k}. \tag{25}$$

There exists the obvious identity

$$(\mathbf{E} - \mathbf{Q}(\boldsymbol{\theta}))\cdot\boldsymbol{\theta} = \left(\mathbf{E} - \mathbf{Q}^T\right)\cdot\boldsymbol{\theta} = \mathbf{0} \Longrightarrow (\mathbf{a} - \mathbf{a}')\cdot\boldsymbol{\theta} = 0, \qquad \mathbf{a}' = \mathbf{Q}\cdot\mathbf{a}.$$

Taking into account this identity and expression (25) one can get

$$(\mathbf{E} - \mathbf{Q}(\boldsymbol{\theta}))\cdot\mathbf{M} = -\frac{\partial F}{\partial(\mathbf{k}\cdot\boldsymbol{\theta})}\mathbf{k}\times\boldsymbol{\theta}.$$

Multiplying this equality by the vector **k** we shall obtain

$$(\mathbf{k} - \mathbf{k}')\cdot\mathbf{M} = \mathbf{0}. \tag{26}$$

For the isotropic potential equality (26) holds good for any vector **a**. Sometimes equality (26) is very important — see, for example, section 4.

2.4 The perturbation method on the set of properly orthogonal tensors

Any turn-tensors must be subjected to restrictions

$$\mathbf{Q} \cdot \mathbf{Q}^{\mathsf{T}} = \mathbf{Q}^{\mathsf{T}} \cdot \mathbf{Q} = \mathbf{E}, \qquad \det \mathbf{Q} = +1. \tag{27}$$

This means that the perturbed tensor of turn $\mathbf{Q}_\varepsilon$ must be subjected to conditions (27) as well. In contrast with this the vector of turn has no restrictions like (27). Because of this the perturbed vector of turn can be defined in the simplest form

$$\boldsymbol{\theta}_\varepsilon = \boldsymbol{\theta} + \varepsilon \boldsymbol{\varphi}, \qquad |\varepsilon| \ll 1, \tag{28}$$

where the vector $\boldsymbol{\varphi}$ is called the first variation of the vector of turn. The perturbed tensor of turn can be found by a usual way

$$\mathbf{Q}_\varepsilon = \exp \mathbf{R}_\varepsilon = \exp\left(\boldsymbol{\theta}_\varepsilon \times \mathbf{E}\right). \tag{29}$$

Equations (27) are satisfied by the tensor $\mathbf{Q}_\varepsilon$ for arbitrary vector $\boldsymbol{\theta}_\varepsilon$. We shall consider the parameter ε as an independent variable. In such case it is possible to introduce the left $\boldsymbol{\eta}_\varepsilon$ and the right $\boldsymbol{\zeta}_\varepsilon$ velocities of perturbation

$$\frac{\partial}{\partial \varepsilon} \mathbf{Q}_\varepsilon = \boldsymbol{\eta}_\varepsilon \times \mathbf{Q}_\varepsilon, \qquad \frac{\partial}{\partial \varepsilon} \mathbf{Q}_\varepsilon = \mathbf{Q}_\varepsilon \times \boldsymbol{\zeta}_\varepsilon, \qquad \boldsymbol{\eta}_\varepsilon = \mathbf{Q}_\varepsilon \cdot \boldsymbol{\zeta}_\varepsilon. \tag{30}$$

The perturbed angular velocities can be found from the Poisson equations

$$\dot{\mathbf{Q}}_\varepsilon = \boldsymbol{\omega}_\varepsilon \times \mathbf{Q}_\varepsilon, \qquad \dot{\mathbf{Q}}_\varepsilon = \mathbf{Q}_\varepsilon \times \boldsymbol{\Omega}_\varepsilon, \qquad \boldsymbol{\omega}_\varepsilon = \mathbf{Q}_\varepsilon \cdot \boldsymbol{\Omega}_\varepsilon. \tag{31}$$

The conditions of integrability for system (30), (31) can be written in the form

$$\frac{\partial}{\partial \varepsilon} \boldsymbol{\omega}_\varepsilon = \dot{\boldsymbol{\eta}}_\varepsilon + \boldsymbol{\eta}_\varepsilon \times \boldsymbol{\omega}_\varepsilon, \qquad \frac{\partial}{\partial \varepsilon} \boldsymbol{\Omega}_\varepsilon = \dot{\boldsymbol{\zeta}}_\varepsilon - \boldsymbol{\zeta}_\varepsilon \times \boldsymbol{\Omega}_\varepsilon. \tag{32}$$

For the velocities of perturbation we have the expressions that are analogous to equations (19)

$$\boldsymbol{\eta}_\varepsilon = \mathbf{Z}^{-1}\left(\boldsymbol{\theta}_\varepsilon\right) \cdot \frac{\partial}{\partial \varepsilon} \boldsymbol{\theta}_\varepsilon = \mathbf{Z}_\varepsilon^{-1} \cdot \boldsymbol{\varphi}, \qquad \boldsymbol{\zeta}_\varepsilon = \mathbf{Z}_\varepsilon^{-\mathsf{T}} \cdot \boldsymbol{\varphi}. \tag{33}$$

The perturbed angular velocities can be found by means of expressions

$$\boldsymbol{\omega}_\varepsilon = \mathbf{Z}_\varepsilon^{-1} \cdot \dot{\boldsymbol{\theta}}_\varepsilon, \qquad \boldsymbol{\Omega}_\varepsilon = \mathbf{Z}_\varepsilon^{-\mathsf{T}} \cdot \dot{\boldsymbol{\theta}}_\varepsilon.$$

If an unperturbed vector $\boldsymbol{\theta}$ does not depend on time (a state of equilibrium), then

$$\boldsymbol{\omega}_\varepsilon = \varepsilon \mathbf{Z}_\varepsilon^{-1} \cdot \dot{\boldsymbol{\varphi}}, \qquad \boldsymbol{\Omega}_\varepsilon = \varepsilon \mathbf{Z}_\varepsilon^{-\mathsf{T}} \cdot \dot{\boldsymbol{\varphi}}. \tag{34}$$

Let there be given the function $f(\varepsilon, t)$. The quantity

$$f^*(t) = \left[\partial f(\varepsilon, t) / \partial \varepsilon\right]_{\varepsilon=0} \tag{35}$$

is called the first variation of the function $f(\varepsilon, t)$. For the first variation of the turn-tensor and of the velocities of perturbation we have

$$\mathbf{Q}^* = \boldsymbol{\eta}_0 \times \mathbf{Q}_0, \qquad \boldsymbol{\eta}_0 = \mathbf{Z}_0^{-1} \cdot \boldsymbol{\varphi}, \qquad \boldsymbol{\omega}^* = \dot{\boldsymbol{\eta}}_0 + \boldsymbol{\eta}_0 \times \boldsymbol{\omega}_0, \tag{36}$$

where the subscripts 0 marks the unperturbed state, $\boldsymbol{\eta}_0 = \boldsymbol{\eta}_\varepsilon \,|_{\varepsilon=0}$.

For the right quantities the next expressions are valid

$$\mathbf{Q}^* = \mathbf{Q}_0 \times \boldsymbol{\zeta}_0, \qquad \boldsymbol{\zeta}_0 = \mathbf{Z}_0^{-T} \cdot \boldsymbol{\varphi}, \qquad \boldsymbol{\Omega}^* = \dot{\boldsymbol{\zeta}}_0 - \boldsymbol{\zeta}_0 \times \boldsymbol{\Omega}_0. \tag{37}$$

If the perturbations are superposed on a state of equilibrium, then $\boldsymbol{\omega}_0 = \boldsymbol{\Omega}_0 = \mathbf{0}$.

Let us write down the formulas for the first variation of modulus of the vector of turn

$$\theta^* = \frac{1}{\theta_0}\boldsymbol{\theta}_0 \cdot \boldsymbol{\varphi} = \frac{1}{\theta_0}\boldsymbol{\theta}_0 \cdot \boldsymbol{\eta}_0 = \frac{1}{\theta_0}\boldsymbol{\theta}_0 \cdot \boldsymbol{\zeta}_0. \tag{38}$$

3 The equations of motion of the rigid body oscillator

Let us consider a rigid body with a fixed point O. The body is supposed to be clamped in an elastic foundation, which is resisting to any turn of the body. The position of the body, in which the elastic foundation is undeformed, we shall choose as the reference position. The tensor of inertia with respect to the fixed point O of the body will be denoted as

$$\mathbf{A} = A_1 \mathbf{d}_1 \otimes \mathbf{d}_1 + A_2 \mathbf{d}_2 \otimes \mathbf{d}_2 + A_3 \mathbf{d}_3 \otimes \mathbf{d}_3, \tag{39}$$

where $A_i > 0$ are the principal moments of inertia and the vectors $\mathbf{d}_i$ are the principal axes of the inertia tensor. Of course the tensor $\mathbf{A}$ can be represented in terms of arbitrary basis $\mathbf{e}_i$

$$\mathbf{d}_i = \alpha_i^m \mathbf{e}_m, \qquad \mathbf{A} = A^{mn} \mathbf{e}_m \otimes \mathbf{e}_n, \qquad A^{mn} = \sum_{i=1}^{3} \alpha_i^m \alpha_i^n A_i.$$

If the body has the axis of symmetry $\mathbf{k}$, then the inertia tensor will be transversally isotropic

$$\mathbf{A} = A_1 (\mathbf{E} - \mathbf{k} \otimes \mathbf{k}) + A_3 \mathbf{k} \otimes \mathbf{k}, \qquad \mathbf{d}_3 = \mathbf{k}, \qquad A_1 = A_2. \tag{40}$$

The position of the body at the instant t we shall call the actual position of the body. A turn of the body can be defined by the turn-tensor $\mathbf{P}(t)$ or by the vector of turn $\boldsymbol{\theta}(t)$

$$\mathbf{P}(t) = \mathbf{Q}(\boldsymbol{\theta}(t)).$$

The tensor of inertia $\mathbf{A}^{(t)}$ in the actual position is determined by the formula

$$\mathbf{A}^{(t)} = \mathbf{P}(t) \cdot \mathbf{A} \cdot \mathbf{P}^T(t). \tag{41}$$

If the tensor $\mathbf{A}$ is transversally isotropic, then one can write down

$$\mathbf{A}^{(t)} = A_1 (\mathbf{E} - \mathbf{k}' \otimes \mathbf{k}') + A_3 \mathbf{k}' \otimes \mathbf{k}', \qquad \mathbf{k}' = \mathbf{P} \cdot \mathbf{k}. \tag{42}$$

A kinetic moment of the body can be expressed in two forms. In terms of the left angular velocity

$$\mathbf{L} = \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{P}^T \cdot \boldsymbol{\omega} = A_1 \boldsymbol{\omega} + (A_3 - A_1)(\mathbf{k}' \cdot \boldsymbol{\omega})\, \mathbf{k}'. \tag{43}$$

Here the first sign of equality concerns to a general case, the second sign of equality is applied to the transversally isotropic tensor of inertia only. In terms of the right angular velocity the kinetic moment has the form

$$\mathbf{L} = \mathbf{P} \cdot \mathbf{A} \cdot \boldsymbol{\Omega} = \mathbf{P} \cdot [A_1 \boldsymbol{\Omega} + (A_3 - A_1)(\mathbf{k} \cdot \boldsymbol{\Omega})\mathbf{k}]. \tag{44}$$

Let us note that

$$\mathbf{k}' \cdot \boldsymbol{\omega} = \mathbf{k} \cdot \mathbf{P}^T \cdot \boldsymbol{\omega} = \mathbf{k} \cdot \boldsymbol{\Omega}. \tag{45}$$

An external torque $\mathbf{M}$ acting on the body can be represented in the form

$$\mathbf{M} = \mathbf{M}_e + \mathbf{M}_{ext},$$

where $\mathbf{M}_e$ is a reaction of the elastic foundation and M_{ext} is an additional external torque. The elastic torque $\mathbf{M}_e$ is supposed to be potential. Besides it is supposed to be positional. In such case we write — see equation (23)

$$\mathbf{M}_e = -\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \frac{dU(\boldsymbol{\theta})}{d\boldsymbol{\theta}}, \tag{46}$$

where the scalar function $U(\boldsymbol{\theta})$ will be called an elastic energy. In what follows the elastic foundation is supposed to be transversally isotropic. This means that the elastic torque can be represented in form (25)

$$\mathbf{M}_e(\boldsymbol{\theta}) = -C\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right)\boldsymbol{\theta} - D\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right)\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \mathbf{k}, \tag{47}$$

where the unit vector $\mathbf{k}$ is placed on the axis of isotropy of the body when the elastic foundation is in the undeformed state.

$$C = 2\frac{\partial}{\partial(\theta^2)}U\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right), \qquad D = \frac{\partial}{\partial(\mathbf{k} \cdot \boldsymbol{\theta})}U\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right). \tag{48}$$

Let us show one of possible expressions of the elastic energy

$$U = \frac{1}{2}\frac{\alpha^2 c \theta^2}{\alpha^2 - \theta^2 + (\mathbf{k} \cdot \boldsymbol{\theta})^2} + \frac{1}{2}\frac{\beta^2 (d - c)(\mathbf{k} \cdot \boldsymbol{\theta})^2}{\beta^2 - (\mathbf{k} \cdot \boldsymbol{\theta})^2}, \tag{49}$$

where $\alpha^2 > 0$, $\beta^2 > 0$, $c > 0$ and $d > 0$ are the constant parameters and also parameters c and d are called the bending rigidness and torsional rigidness of the elastic foundation respectively.

If the parameters α^2 and β^2 tend to the infinity, then we shall get the simplest form of the elastic potential

$$U = \frac{1}{2}c\left(\theta^2 - (\mathbf{k} \cdot \boldsymbol{\theta})^2\right) + \frac{1}{2}d(\mathbf{k} \cdot \boldsymbol{\theta})^2. \tag{50}$$

In this case expression (47) takes the form

$$\mathbf{M}_e(\boldsymbol{\theta}) = -c\boldsymbol{\theta} - (d - c)\,\mathbf{k} \cdot \boldsymbol{\theta}\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \mathbf{k}. \tag{51}$$

For the external torque $\mathbf{M}_{ext}$ let us accept the expression

$$\mathbf{M}_{ext} = -\mathbf{Z}^{T}(\theta) \cdot \frac{dV(\theta)}{d\theta} + \mathbf{M}_{ex}, \qquad (52)$$

where the first term describes the potential part of the external torque. The second law of dynamics of Euler can be represented in two equivalent forms. In terms of the left angular velocity it takes the form

$$\left[\mathbf{P}(\theta) \cdot \mathbf{A} \cdot \mathbf{P}^{T}(\theta) \cdot \omega\right]^{\cdot} + \mathbf{Z}^{T}(\theta) \cdot \frac{d(U+V)}{d\theta} = \mathbf{M}_{ex}. \qquad (53)$$

To this equation we have to add the left Poisson equation in form (15)

$$\dot{\theta} = \omega - \frac{1}{2}\theta \times \omega + \frac{1-g}{\theta^2}\theta \times (\theta \times \omega). \qquad (54)$$

System of equations (53) and (54) gives to us a general model of the rigid body oscillator. In terms of the right angular velocity this model can be represented in the form

$$\mathbf{A} \cdot \dot{\mathbf{\Omega}} + \mathbf{\Omega} \times \mathbf{A} \cdot \mathbf{\Omega} + \mathbf{Z}(\theta) \cdot \frac{d(U+V)}{d\theta} = \mathbf{P}^{T}(\theta) \cdot \mathbf{M}_{ex}, \qquad (55)$$

$$\dot{\theta} = \mathbf{\Omega} + \frac{1}{2}\theta \times \mathbf{\Omega} + \frac{1-g}{\theta^2}\theta \times (\theta \times \mathbf{\Omega}). \qquad (56)$$

It is important that the model of rigid body oscillator is represented in terms of natural variables: the vector of turn and the vector of angular velocity. Besides significant merit of stated above equations is that they contain the first derivatives of the unknown vectors only. Thus it is possible to use standard methods of the numerical analysis.

The rest of the paper deals with applications of the derived equations.

4 The stability of equilibrium state of rigid body oscillator under the action of the follower torque. Paradox of Nikolai

Let us consider the classical problem that was investigated by E.L.Nikolai [3]. Later it was studied by many authors — see, for example, [4], [5], where another references can be found.

The inertia tensor of the body is supposed to be transversally isotropic and is defined by expression (40). An external torque is defined by the next expression

$$\mathbf{M}_{ex} = L\mathbf{P}(\theta) \cdot \mathbf{k}, \qquad L = \text{const}, \qquad (57)$$

where the unit vector $\mathbf{k}$ is placed on the axis of symmetry of the body in the reference position when the elastic foundation is undeformed.

Accepting the stated above assumptions we are able to write down equations (55) and(56) in the next form.

$$A_1\dot{\mathbf{\Omega}} + (A_3 - A_1)\left(\mathbf{k} \cdot \dot{\mathbf{\Omega}}\right)\mathbf{k} - (A_3 - A_1)(\mathbf{k} \cdot \mathbf{\Omega})\mathbf{k} \times \mathbf{\Omega} + C\theta + D\mathbf{Z}^{T}(\theta) \cdot \mathbf{k} = L\mathbf{k}, \quad (58)$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\Omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\Omega}), \qquad g = \frac{\theta \sin\theta}{2(1-\cos\theta)}, \tag{59}$$

where the functions C and D are defined by expressions (48). It is easy to find the equilibrium solution of system of equations (58) and (59)

$$\boldsymbol{\theta} = \theta \mathbf{k}, \qquad \theta = const, \qquad \boldsymbol{\Omega} = 0. \tag{60}$$

Substituting (60) into system (58)–(59) we shall get the scalar equation

$$C\left(\theta^2, \theta\right)\theta + D\left(\theta^2, \theta\right) = L. \tag{61}$$

If the elastic energy has form (50), then equation (61) takes the linear form

$$C\left(\theta^2, \theta\right) = c, D\left(\theta^2, \theta\right) = (d-c)\,\mathbf{k} \cdot \boldsymbol{\theta} \Longrightarrow \boldsymbol{\theta} = L\mathbf{k}/d. \tag{62}$$

In order to investigate a stability of the solution of equation (61) we shall use the method of superposition of small perturbations on the state of equilibrium. To this end let us consider the perturbed quantities

$$\boldsymbol{\theta}_\varepsilon = \theta\mathbf{k} + \varepsilon\boldsymbol{\varphi}(t), \qquad \boldsymbol{\Omega}_\varepsilon = \varepsilon\boldsymbol{\eta}, \tag{63}$$

where θ is the solution of (61).

Now we have to write down perturbed equations (58) and (59). For this it is sufficiently to provide the vectors $\boldsymbol{\theta}$ and $\boldsymbol{\Omega}$ in these equations by the subscripts ε. After that it is necessary to differentiate these equations with respect to ε and to accept $\varepsilon = 0$. As the result we shall get equations in variations $\boldsymbol{\varphi}$ and $\boldsymbol{\eta}$.

For the sake of simplicity let us consider case (62). In such case perturbed equations (58) and (59) take the form

$$A_1\dot{\boldsymbol{\Omega}}_\varepsilon + (A_3 - A_1)\left(\mathbf{k} \cdot \dot{\boldsymbol{\Omega}}_\varepsilon\right)\mathbf{k} - (A_3 - A_1)(\mathbf{k} \cdot \boldsymbol{\Omega}_\varepsilon)\,\mathbf{k} \times \boldsymbol{\Omega}_\varepsilon + c\boldsymbol{\theta}_\varepsilon +$$

$$+ (d-c)\,\mathbf{k} \cdot \boldsymbol{\theta}_\varepsilon \mathbf{Z}^T(\boldsymbol{\theta}_\varepsilon) \cdot \mathbf{k} = L\mathbf{k}, \tag{64}$$

$$\dot{\boldsymbol{\theta}}_\varepsilon = \boldsymbol{\Omega}_\varepsilon + \frac{1}{2}\boldsymbol{\theta}_\varepsilon \times \boldsymbol{\Omega}_\varepsilon + \frac{1-g_\varepsilon}{\theta_\varepsilon^2}\boldsymbol{\theta}_\varepsilon \times (\boldsymbol{\theta}_\varepsilon \times \boldsymbol{\Omega}_\varepsilon), g_\varepsilon = \frac{\theta_\varepsilon \sin\theta_\varepsilon}{2(1-\cos\theta_\varepsilon)}. \tag{65}$$

Expressions (63) take the form

$$\boldsymbol{\theta}_\varepsilon = \frac{L}{d}\mathbf{k} + \varepsilon\boldsymbol{\varphi}, \qquad \boldsymbol{\Omega}_\varepsilon = \varepsilon\boldsymbol{\eta}, \qquad \boldsymbol{\theta}_\varepsilon \times \mathbf{k} = \varepsilon\boldsymbol{\varphi} \times \mathbf{k}. \tag{66}$$

The equations in variations can be represented as

$$A_1\dot{\boldsymbol{\eta}} + (A_3 - A_1)(\mathbf{k} \cdot \dot{\boldsymbol{\eta}})\,\mathbf{k} + c\boldsymbol{\varphi} + (d-c)(\mathbf{k} \cdot \boldsymbol{\varphi})\,\mathbf{k} +$$

$$+L\left(1 - \frac{c}{d}\right)\left[\frac{1}{2}\boldsymbol{\varphi} \times \mathbf{k} + \frac{1-g}{\theta}(\boldsymbol{\varphi} - \mathbf{k} \cdot \boldsymbol{\varphi}\mathbf{k})\right] = 0,$$

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{\eta} + \frac{1}{2}\frac{L}{d}\mathbf{k} \times \boldsymbol{\eta} - (1-g)(\boldsymbol{\eta} - (\mathbf{k} \times \boldsymbol{\eta})\,\mathbf{k}).$$

These equations can be rewritten in more simple form with the help of substitution

$$\boldsymbol{\eta} = \zeta\mathbf{k} + \mathbf{y}, \qquad \mathbf{y}\cdot\mathbf{k} = 0; \qquad \boldsymbol{\varphi} = \gamma\mathbf{k} + \boldsymbol{\psi}, \qquad \boldsymbol{\psi}\cdot\mathbf{k} = 0. \tag{67}$$

After some transformations one can write

$$A_3\ddot{\gamma} + d\gamma = 0, \qquad \zeta = \dot{\gamma}, \tag{68}$$

$$A_1\ddot{\boldsymbol{\psi}} + \left[c\left(g^2 + \frac{L^2}{4d^2}\right) - \frac{L^2}{4d} + (1-g)\,gd\right]\boldsymbol{\psi} + \frac{L}{2}\mathbf{k}\times\boldsymbol{\psi} = \mathbf{0}, \tag{69}$$

where

$$g = \frac{\theta\sin\theta}{2\,(1-\cos\theta)}, \qquad \theta = \frac{L}{d}.$$

If the quantity $|L|/d$ is small, i.e. $|L|/d \ll 1$, then equation (69) can be rewritten as

$$A_1\ddot{\boldsymbol{\psi}} + \left[c + \frac{(c-2d)\,L^2}{12d^2}\right]\boldsymbol{\psi} + \frac{L}{2}\mathbf{k}\times\boldsymbol{\psi} = 0. \tag{70}$$

Let us look for a particular solution of these equation in the form

$$\boldsymbol{\psi} = \mathbf{a}\exp(pt), \qquad \mathbf{a} = \mathrm{const}, \qquad \mathbf{a}\cdot\mathbf{k} = 0.$$

For the vector $\mathbf{a}$ we have the system

$$\left[A_1\left(p^2 + c + \frac{(c-2d)\,L^2}{12d^2}\right)\mathbf{E}_* + \frac{L}{2}\mathbf{k}\times\mathbf{E}_*\right]\cdot\mathbf{a} = \mathbf{0}, \quad \mathbf{E}_* = \mathbf{E} - \mathbf{k}\otimes\mathbf{k}.$$

The determinant of this system must be equal to zero

$$\left[A_1\left(p^2 + c + \frac{(c-2d)\,L^2}{12d^2}\right)\right]^2 + \frac{L^2}{4} = 0.$$

It is easy to see that at least one root of this equation has a positive real part. From this it follows that the solution of equation (70) infinitely increases. This means that the state of equilibrium (62) or (61) is unstable for arbitrarily small quantity of external twisting moment L. This phenomenon is well known under the name of paradox of Nikolai.

From the pure theoretical point of view it is no wonder that the state of equilibrium is unstable. However, from the practical point of view the situation is very disagreeable. Really, if the external torque is small, then it is supposed that the linear theory is valid. In this case system of equations (58) and (59) can be rewritten in the form of equation

$$A_1\ddot{\boldsymbol{\theta}} + (A_3 - A_1)\,(\mathbf{k}\cdot\ddot{\boldsymbol{\theta}})\mathbf{k} + c\boldsymbol{\theta} + (d-c)\,(\mathbf{k}\cdot\boldsymbol{\theta})\,\mathbf{k} = L\mathbf{k}.$$

The solution of this equation has a small norm if the torque L and the norm of initial conditions are small. Namely this way is used in the most of applied investigations. There was no doubts that such approach is quite accurate. However, as it was shown above, if we take into account the small quantities of the second order, then the solution will be unstable. Is it really so? It is well-known fact [6] that the equations in variations may give a faulty result in some cases. This means that in doubtful cases the nonlinear analysis have to be used.

5 Nonlinear analysis and rigorous justification of the paradox of Nikolai

Let us consider the external torque of the kind

$$\mathbf{M}_{ex} = \gamma L \left(l_1 \mathbf{k} + l_2 \mathbf{P} \cdot \mathbf{k} \right), \qquad \gamma = \left(l_1^2 + l_2^2 + 2 l_1 l_2 \mathbf{k} \cdot \mathbf{P} \cdot \mathbf{k} \right)^{-\frac{1}{2}}. \tag{71}$$

If $l_1 = 1$, $l_2 = 0$, then $\mathbf{M}_{ex}$ is a dead torque; if $l_1 = 0$, $l_2 = 1$, then $\mathbf{M}_{ex}$ is a followed (tangential) torque; if $l_1 = l_2 = 1$, then $\mathbf{M}_{ex}$ is a semitangential torque. For the elastic torque let us accept expression (47) , where $C\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right)$ and $D\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right)$ are the functions of a general kind. The tensor of inertia is supposed to be transversally isotropic with the axis of symmetry $\mathbf{k}$.

For the vector of kinetic moment we have formulas (43) and (44). Let us write down the equation of the energy balance when the external torque is defined by expression (71)

$$\dot{\varepsilon} = \gamma L \left(l_1 \mathbf{k} \cdot \boldsymbol{\omega} + l_2 \mathbf{k} \cdot \boldsymbol{\Omega} \right), \qquad \varepsilon = \frac{1}{2} A_1 \omega^2 + \frac{1}{2} \left(A_3 - A_1 \right) \left(\mathbf{k} \cdot \boldsymbol{\Omega} \right)^2 + U(\boldsymbol{\theta}). \tag{72}$$

From (72) it follows

$$\varepsilon - \varepsilon_0 = L \int\limits_0^t \gamma(\tau)\, \mathbf{k} \cdot \left(l_1 \boldsymbol{\omega}(\tau) + l_2 \boldsymbol{\Omega}(\tau) \right) d\tau. \tag{73}$$

If the integral in the right side of equation (73) is bounded for all t, then for small $|L|$ the energy ε is close to the value of the initial energy ε_0. In such a case the stability is possible. If integral (73) is infinitely increasing, then we have the accumulation of energy in the system and the stability is impossible for arbitrarily small $|L|$.

Let us write the equation of motion in two forms

$$\left[A_1 \boldsymbol{\omega} + \left(A_3 - A_1 \right) \left(\boldsymbol{\omega} \cdot \mathbf{k}' \right) \mathbf{k}' \right]^{\cdot} + C \boldsymbol{\theta} + D \mathbf{Z}^T \cdot \mathbf{k} = \gamma L \left(l_1 \mathbf{k} + l_2 \mathbf{k}' \right), \tag{74}$$

$$\left[A_1 \boldsymbol{\Omega} + \left(A_3 - A_1 \right) \left(\boldsymbol{\Omega} \cdot \mathbf{k} \right) \mathbf{k} \right]^{\cdot} + \left(A_3 - A_1 \right) \left(\mathbf{k} \cdot \boldsymbol{\Omega} \right) \boldsymbol{\Omega} \times \mathbf{k} + \\ + C \boldsymbol{\theta} + D \mathbf{Z} \cdot \mathbf{k} = \gamma L \left(l_1 \mathbf{P}^T \cdot \mathbf{k} + l_2 \mathbf{k} \right), \tag{75}$$

where $\boldsymbol{\omega} \cdot \mathbf{k}' = \boldsymbol{\Omega} \cdot \mathbf{k}$, $\mathbf{k}' = \mathbf{P} \cdot \mathbf{k}$.

Equations (74) and (75) are equivalent. Nevertheless from them the nontrivial result can be found. Subtracting equation (75) from equation (74) one can get

$$\left[A_1 \left(\boldsymbol{\omega} - \boldsymbol{\Omega} \right) + \left(A_3 - A_1 \right) \left(\boldsymbol{\Omega} \cdot \mathbf{k} \right) \left(\mathbf{k}' - \mathbf{k} \right) \right]^{\cdot} + \\ + \left(A_3 - A_1 \right) \left(\mathbf{k} \cdot \boldsymbol{\Omega} \right) \mathbf{k} \times \boldsymbol{\Omega} + D \boldsymbol{\theta} \times \mathbf{k} = \gamma L \left[\left(l_1 - l_2 \right) \mathbf{k} + l_2 \mathbf{k}' - l_1 \mathbf{P}^T \cdot \mathbf{k} \right].$$

Multiplying this equation by the vector $\mathbf{k}$ we shall obtain the next equation

$$\left[A_1 \left(\boldsymbol{\omega} - \boldsymbol{\Omega} \right) \cdot \mathbf{k} + \left(A_1 - A_3 \right) \mathbf{k} \cdot \boldsymbol{\Omega} \left(1 - \cos\vartheta \right) \right]^{\cdot} = \\ = \gamma L \left(l_1 - l_2 \right) \left(1 - \cos\vartheta \right), \tag{76}$$

where $\cos\vartheta = \mathbf{k}\cdot\mathbf{k}' = \mathbf{k}\cdot\mathbf{P}\cdot\mathbf{k}$.

Let us note that equation (76) does not contain the characteristics of the elastic foundation. Equation (76) can be rewritten in another form. From equations (19) and (20) it follows

$$\boldsymbol{\omega} - \boldsymbol{\Omega} = \left(\mathbf{Z}^{-1} - \mathbf{Z}^{-T}\right)\cdot\dot{\boldsymbol{\theta}} = 2\frac{1-\cos\theta}{\theta^2}\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}. \tag{77}$$

The vector of turn $\boldsymbol{\theta}$ can be represented in the form of the composition

$$\boldsymbol{\theta} = x\mathbf{k} + \mathbf{y}, \mathbf{y}\cdot\mathbf{k} = 0, \mathbf{y} = y\left(t\right)\mathbf{Q}\left(\psi\left(t\right)\mathbf{k}\right)\cdot\mathbf{m}, \tag{78}$$
$$\mathbf{m}\cdot\mathbf{k} = 0, |\mathbf{m}| = 1, \theta^2 = x^2 + y^2.$$

One can prove the formulas

$$\mathbf{k}\cdot\left(\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}\right) = \mathbf{k}\cdot\left(\mathbf{y}\times\dot{\mathbf{y}}\right) = \dot{\psi}y^2, \quad 1-\cos\vartheta = \frac{y^2\left(1-\cos\theta\right)}{\theta^2}. \tag{79}$$

Taking into account relations (77), (78) and (79) equation (76) can be rewritten in the form

$$\left[\left(1-\cos\vartheta\right)F\right]^{\cdot} = \gamma L\left(l_1 - l_2\right)\left(1-\cos\vartheta\right), \tag{80}$$

where

$$F = 2A_1\dot{\psi} + \left(A_1 - A_3\right)\mathbf{k}\cdot\boldsymbol{\Omega}.$$

Equality (80) was derived by another way and was shown to the author in the private talk by Dr. A. Krivtsov. In fact equality (80) is due to the existence of property (26) for the elastic torque. Let us note that the right side of equation (80) has the constant sign, which is defined by the sign of the number $L\left(l_1 - l_2\right)$. Let us suppose that $L\left(l_1 - l_2\right) > 0$. In such a case let us choose the initial conditions such that $F\,|_{t=0} > 0$. Equality (80) shows to us that the function $F(t)$ tends to infinity as $t \to \infty$. This means that the body will have an infinitely big velocity of precession $\dot{\psi}$, i.e. state of equilibrium (61) or (62) is unstable for arbitrarily small value of twisting torque and for any transversally isotropic elastic foundation. Therefore the analysis on the base of the equations in variations gives the right result. The paradox of Nikolai is due to an accumulation of energy in the system.

6 The simplest rigid body oscillator. The total integrability of the basic equations

Let us consider the simplest case of the rigid body oscillator. For this end let us accept the next restrictions

$$\mathbf{A} = A\mathbf{E}, \qquad U = u\left(\theta^2\right), \qquad \frac{d}{d\boldsymbol{\theta}}U = 2u'\left(\theta^2\right)\boldsymbol{\theta} = c\left(\theta^2\right)\boldsymbol{\theta}. \tag{81}$$

In addition let us introduce the torque of friction in the form

$$\mathbf{M}_{ex} = -b\boldsymbol{\omega}, \qquad b = \mathrm{const} \geqslant 0. \tag{82}$$

In such a case basic equations (55) and (56) can be written down in the form

$$A\dot{\boldsymbol{\Omega}} + b\boldsymbol{\Omega} + c\left(\theta^2\right)\boldsymbol{\theta} = \mathbf{0}, \tag{83}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\theta}\times\boldsymbol{\Omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\boldsymbol{\Omega}), \qquad g = \frac{\theta\sin\theta}{2(1-\cos\theta)}. \tag{84}$$

It is seen that even in this simplest case the basic system is rather complicated. The system can be simplified only in the case of the plane oscillations when

$$\boldsymbol{\omega} = \boldsymbol{\Omega} = \dot{\boldsymbol{\theta}}, \qquad \boldsymbol{\theta}\times\boldsymbol{\Omega} = \mathbf{0}.$$

If it is so, then system (83) and (84) takes the form

$$A\ddot{\boldsymbol{\theta}} + b\dot{\boldsymbol{\theta}} + c\left(\theta^2\right)\boldsymbol{\theta} = \mathbf{0}; \qquad t = 0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0, \qquad \boldsymbol{\Omega} = \boldsymbol{\Omega}_0, \qquad \boldsymbol{\theta}_0\times\boldsymbol{\Omega}_0 = \mathbf{0}. \tag{85}$$

This system can be investigated without any problems.

Let us discuss system of equations (83) and (84) in a general case. In order to underline the difference between conventional approach and our method let us consider both of them.

6.1 Conventional approach

Let us try to investigate system (83), (84) on the base of application of the Euler angles. The tensor of turn can be represented [1] in the form

$$\mathbf{P}(\boldsymbol{\theta}) = \mathbf{Q}(\psi\mathbf{k})\cdot\mathbf{Q}(\vartheta\mathbf{p})\cdot\mathbf{Q}(\varphi\mathbf{k}) = \mathbf{Q}(\vartheta\mathbf{p}')\cdot\mathbf{Q}(\beta\mathbf{k}), \tag{86}$$

where

$$\beta = \varphi + \psi, \qquad \mathbf{p}' = \mathbf{Q}(\psi\mathbf{k})\cdot\mathbf{p}, \qquad \mathbf{k}\cdot\mathbf{p} = \mathbf{k}\cdot\mathbf{p}' = 0. \tag{87}$$

The left angular velocity is determined by the formula

$$\boldsymbol{\omega} = \left(\dot\psi + \dot\varphi\cos\vartheta\right)\mathbf{k} + \dot\vartheta\mathbf{p}' + \dot\varphi\sin\vartheta\,\mathbf{p}'\times\mathbf{k}. \tag{88}$$

Making use expressions (7), (86), (88) and substituting them into equation (90) one can derive the system

$$A\left(\dot\psi + \dot\varphi\cos\vartheta\right)^{\cdot} + b\left(\dot\psi + \dot\varphi\cos\vartheta\right) + \frac{c\left(\theta^2\right)\theta}{2\sin\theta}\sin\beta\,(1+\cos\vartheta) = 0,$$

$$A\left(\ddot\vartheta + \dot\psi\dot\varphi\sin\vartheta\right) + b\dot\vartheta + \frac{c\left(\theta^2\right)\theta}{2\sin\theta}\sin\vartheta\,(1+\cos\beta) = 0,$$

$$A\left[(\dot\varphi\sin\vartheta)^{\cdot} - \dot\varphi\dot\vartheta\right] + b\dot\varphi\sin\vartheta + \frac{c\left(\theta^2\right)\theta}{2\sin\theta}\sin\beta\sin\vartheta = 0. \tag{89}$$

In addition to this system we have the relations

$$1 + 2\cos\theta = \cos\vartheta + \cos\beta + \cos\vartheta\cos\beta, \beta = \varphi + \psi.$$

It is not so easy to find the total solution of system (89). Let us note that representation (86) is completely admissible. However, there are many another possibilities and the most of them will lead to the complicated equations. If we want to find the best representation, then we have to look for this representation in the process of a solution rather then to guess it a priori. The latter circumstances was underlined in the paper [1].

6.2 The total integrability of the equations of the simplest rigid body oscillator

Multiplying equation (83) by the tensor $\mathbf{P}(\boldsymbol{\theta})$ from the left one can obtain

$$A\dot{\boldsymbol{\omega}} + b\boldsymbol{\omega} + c\left(\theta^2\right)\boldsymbol{\theta} = \mathbf{0}. \tag{90}$$

Here the identity

$$\mathbf{P}\cdot\dot{\boldsymbol{\Omega}} = (\mathbf{P}\cdot\boldsymbol{\Omega})^{\cdot} - \dot{\mathbf{P}}\cdot\boldsymbol{\Omega} = \dot{\boldsymbol{\omega}} - (\mathbf{P}\times\boldsymbol{\Omega})\cdot\boldsymbol{\Omega} = \dot{\boldsymbol{\omega}}$$

was taken into account.

Equation (90) is equivalent to equation (83). However from (83) and (90) the non-trivial result follows

$$A(\boldsymbol{\omega}-\boldsymbol{\Omega})^{\cdot} + b(\boldsymbol{\omega}-\boldsymbol{\Omega}) = \mathbf{0} \Longrightarrow \boldsymbol{\omega}-\boldsymbol{\Omega} = (\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0)\exp\left(-\frac{bt}{A}\right), \tag{91}$$

where $\boldsymbol{\omega}_0$ and $\boldsymbol{\Omega}_0$ are the initial angular velocities. Expression (91) gives to us three integrals. Now it is necessary to consider two cases

$$a)\,\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0=\mathbf{0}, \qquad b)\,\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0=|\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0|\,\mathbf{e}\neq\mathbf{0}.$$

In the first case we deal with the plane vibrations of the oscillator. Really, in the first case from (91) it follows that

$$\boldsymbol{\omega}=\boldsymbol{\Omega} \Longrightarrow \boldsymbol{\Omega}\times\boldsymbol{\theta}=\mathbf{0}.$$

The latter fact follows from (15) and (16). Thus we have equation (85). It is more interesting to investigate the case b). From equations (15) and (16) the next relation can be derived.

$$g(\theta)(\boldsymbol{\omega}-\boldsymbol{\Omega}) = \frac{1}{2}\boldsymbol{\theta}\times(\boldsymbol{\omega}+\boldsymbol{\Omega}).$$

Taking into account integral (91) one can get

$$g(\theta)\exp\left(-\frac{bt}{A}\right)(\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0) = \frac{1}{2}\boldsymbol{\theta}\times(\boldsymbol{\omega}+\boldsymbol{\Omega}).$$

Besides let us take into account the identity

$$\frac{1}{2}\boldsymbol{\theta}\times(\boldsymbol{\omega}+\boldsymbol{\Omega}) = \frac{\sin\theta}{\theta}\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}.$$

The previous expression can be rewritten in the form

$$\frac{2(1-\cos\theta)}{\theta^2}\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}} = (\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0)\exp\left(-\frac{bt}{A}\right). \tag{92}$$

From this equation one more integral follows

$$\boldsymbol{\theta}(t)\cdot(\boldsymbol{\omega}_0-\boldsymbol{\Omega}_0)=\mathbf{0} \Longrightarrow \boldsymbol{\theta}(t)\cdot\mathbf{e}=\mathbf{0}, \tag{93}$$

where the vector $\mathbf{e}$ is the vector $(\boldsymbol{\omega}_0 - \boldsymbol{\Omega}_0) / |\boldsymbol{\omega}_0 - \boldsymbol{\Omega}_0|$. Equation (93) shows that the vector $\boldsymbol{\theta}(t)$ can be represented in the form

$$\boldsymbol{\theta}(t) = \theta(t)\, \mathbf{Q}(\psi \mathbf{e}) \cdot \mathbf{m}, \qquad \mathbf{m} = \boldsymbol{\theta}_0/\theta_0, \qquad \mathbf{m} \cdot \mathbf{e} = 0, \qquad \psi(0) = 0. \tag{94}$$

From this representation it follows

$$\boldsymbol{\theta} \times \dot{\boldsymbol{\theta}} = \dot{\psi}\theta^2 \mathbf{e}. \tag{95}$$

Substituting (95) into (92) one can get

$$\dot{\psi} = \frac{1-\cos\theta_0}{1-\cos\theta(t)}\dot{\psi}_0 \exp\left(-\frac{bt}{A}\right), \qquad \dot{\psi}_0 > 0. \tag{96}$$

Thus if we know the angle of nutation $\theta(t)$ then the angle of precession can be found from (96). Let us derive the equation for the angle θ. For this end let us calculate the right angular velocity

$$\boldsymbol{\Omega} = \frac{\dot{\theta}}{\theta}\boldsymbol{\theta} + \frac{\sin\theta}{\theta}\dot{\psi}\mathbf{e} \times \boldsymbol{\theta} - (1-\cos\theta)\,\dot{\psi}\mathbf{e}. \tag{97}$$

Substituting expression (97) into equation (83) and projecting the obtained equation on the vectors $\boldsymbol{\theta}$, $\mathbf{e}$ and $\mathbf{e} \times \boldsymbol{\theta}$ one can get three scalar equations, where two of them (projections on $\mathbf{e}$ and $\mathbf{e} \times \boldsymbol{\theta}$) will be identities because of equality (96). Projection on the vector $\boldsymbol{\theta}$ gives

$$A\left[\ddot{\theta} - \sin\theta\left(\frac{1-\cos\theta_0}{1-\cos\theta}\right)^2 \left(\dot{\psi}_0\right)^2 \exp\left(-\frac{2bt}{A}\right)\right] + b\dot{\theta} + c\left(\theta^2\right)\theta = 0. \tag{98}$$

If the friction is absent $(b = 0)$, then this equation can be solved in terms of quadratures. The plane motions of the oscillator can be found from equation (98) when $\dot{\psi}_0 = 0$. In a general case equation (98) can be studied by conventional methods of nonlinear mechanics. Let us note that even for small θ equation (98) is nonlinear one.

$$A\ddot{\theta} + b\dot{\theta} + \left[c(0) - A\left(\frac{\theta_0}{\theta}\right)^4 \dot{\psi}_0^2 \exp\left(-\frac{2bt}{A}\right)\right]\theta = 0. \tag{99}$$

In contrast with it for small turns system of equations (83) and (84) can be linearized and we shall get the linear equation

$$A\ddot{\boldsymbol{\theta}} + b\dot{\boldsymbol{\theta}} + c(0)\,\boldsymbol{\theta} = 0. \tag{100}$$

Nonlinear equation (99) can be derived from equation (100) if one take into account that $\theta = |\boldsymbol{\theta}|$. If the friction is absent $(b = 0)$ then equation (98) has an exact solution

$$\theta = \theta_0 = \mathrm{const}, \qquad \dot{\psi} = \dot{\psi}_0 = \mathrm{const}, \qquad \left(\dot{\psi}\right)^2 = \frac{c\left(\theta_0^2\right)\theta_0}{A\sin\theta_0}. \tag{101}$$

This solution is called a regular precession, which will be considered in the next section. If the friction is present, then for the big times equation (98) transforms to equation (85).

Let us compare two described approaches. The first approach is defined by representation (86) of the turn-tensor, where the unit vectors $\mathbf{k}$ and $\mathbf{p}$ $(\mathbf{k} \cdot \mathbf{p} = 0)$ were chosen a priori. This means that for angles ψ, ϑ, φ and the velocities $\dot{\psi}, \dot{\vartheta}, \dot{\varphi}$ we have to provide the arbitrary initial conditions. In other words we have to look for a general solution of the system (89). It is not known if it is possible.

In the second approach the representation of the turn-tensor has a special form

$$\mathbf{P} = \mathbf{Q}(\boldsymbol{\theta}) = \mathbf{Q}[\theta \mathbf{Q}(\psi \mathbf{e}) \cdot \mathbf{m}] = \mathbf{Q}(\psi \mathbf{e}) \cdot \mathbf{Q}(\theta \mathbf{m}) \cdot \mathbf{Q}^{\mathsf{T}}(\psi \mathbf{e}). \tag{102}$$

Here we used representation (94) for the vector of turn and the unit vectors $\mathbf{e}$ and $\mathbf{m}$, which were found in the process of solution. Representation (102) contains only two angles θ and ψ, but the unit vectors $\mathbf{e}$, $\mathbf{m}$ are chosen by a special manner. Representation (86) contains three angles ψ, ϑ and φ but the unit vectors $\mathbf{k}$ and $\mathbf{p}$ can be any orthogonal vectors. Let us accept the relation $\varphi = -\psi$, i.e. $\beta = 0$, in representation (86). In such case system (89) takes the form $(\beta = 0)$

$$A\left[\dot{\psi}(1-\cos\theta)\right]^{\cdot} + b\dot{\psi}(1-\cos\theta) = 0,$$

$$A\left(\ddot{\theta} - \dot{\psi}^2 \sin\theta\right) + b\dot{\vartheta} + c\left(\theta^2\right)\theta = 0,$$

$$A\left[\left(\dot{\psi}\sin\theta\right)^{\cdot} - \dot{\theta}\dot{\psi}\right] + b\dot{\psi}\sin\theta = 0.$$

The first equation of this system gives to us integral (96). The third equation is an identity if we take into account the first equation. At last, the second equation coincides with equation (98). Thus system (89) has a particular solution coinciding with the found above solution. However when using representation (86) this solution does not allow to satisfy all initial conditions since the vectors $\mathbf{k}$ and $\mathbf{p}$ have the preassigned directions.

Let us turn back to equation (98). A general analysis of this equation can be made by means of conventional methods. Because of this there is no need to do it in this paper.

7 The regular precession and the equations in variations

Let us consider the body with the transversally isotropic tensor of inertia. The elastic foundation is supposed to be transversally isotropic as well. The equations of motion are given by expressions (53), (54) and expression (47) for the elastic torque.

One can write down

$$\left[A_1 \boldsymbol{\omega} + (A_3 - A_1)(\mathbf{k}' \cdot \boldsymbol{\omega})\mathbf{k}'\right]^{\cdot} + C\boldsymbol{\theta} + D\mathbf{Z}^{\mathsf{T}} \cdot \mathbf{k} = \mathbf{0}, \qquad \mathbf{k}' = \mathbf{P} \cdot \mathbf{k}, \tag{103}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1-g}{\theta^2}\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega}), \tag{104}$$

where the function C and D are defined by expressions (48).

A particular solution of system (103), (104) can be represented in the form

$$\boldsymbol{\theta} = \vartheta \mathbf{p}', \qquad \mathbf{p}' = \mathbf{Q}(\psi \mathbf{k}) \cdot \mathbf{p}, \qquad \mathbf{P} = \mathbf{Q}(\vartheta \mathbf{p}'), \qquad \mathbf{p} \cdot \mathbf{k} = 0. \tag{105}$$

Motion (105) is called a regular precession if the restrictions

$$\vartheta = \text{const}, \qquad \dot{\psi} = \text{const} \tag{106}$$

hold good. The left angular velocity is defined in such case by the formula

$$\boldsymbol{\omega} = \mathbf{Q}(\psi \mathbf{k}) \cdot \boldsymbol{\omega}_0, \qquad \boldsymbol{\omega}_0 = \dot{\psi}\left[(1-\cos\vartheta)\,\mathbf{k} + \sin\vartheta \mathbf{k}\times\mathbf{p}\right] = \text{const}. \tag{107}$$

We see that the vector $\boldsymbol{\omega}$ is a precession of the vector $\boldsymbol{\omega}_0$ around the axis $\mathbf{k}$. Also there are properties

$$\boldsymbol{\theta}\cdot\boldsymbol{\omega} = \boldsymbol{\theta}\cdot\boldsymbol{\Omega} = 0, \qquad \mathbf{k}\cdot\boldsymbol{\theta} = 0.$$

This means that the vector of turn is orthogonal to the vector of angular velocity. In addition let us accept the restriction

$$D\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right)|_{\mathbf{k}\cdot\boldsymbol{\theta}=0} = \frac{\partial}{\partial(\mathbf{k}\cdot\boldsymbol{\theta})} U\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right)|_{\mathbf{k}\cdot\boldsymbol{\theta}=0} = 0,$$

which is satisfied for the most kinds of elastic energy. After substitution (105)–(107) into equations (103), (104) we shall get the identities if the equality

$$\dot{\psi}^2 = \frac{C\left(\vartheta^2, 0\right)\vartheta}{\sin\vartheta\left[A_3\left(1-\cos\vartheta\right) + A_1\cos\vartheta\right]} \tag{108}$$

is valid. If $A_1 = A_3 = A$, then we have expression (101). Thus expressions (105)–(108) give to us the exact solution of system (103)–(104).

Now we must investigate a stability of solution (105)–(108). Generally it is rather cumbersome process. In order to simplify it let us accept

$$A = A_1 = A_3, \qquad D\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right) = 0, \qquad C\left(\theta^2, \mathbf{k}\cdot\boldsymbol{\theta}\right) = c = \text{const}. \tag{109}$$

This means that the tensor of inertia and the elastic foundation are supposed to be isotropic. Under these assumptions perturbed equations of motion (103)–(104) take the form

$$A\dot{\boldsymbol{\omega}}_\varepsilon + c\boldsymbol{\theta}_\varepsilon = \mathbf{0},$$
$$\dot{\boldsymbol{\theta}}_\varepsilon = \boldsymbol{\omega}_\varepsilon - \frac{1}{2}\boldsymbol{\theta}_\varepsilon\times\boldsymbol{\omega}_\varepsilon + \frac{1-g_\varepsilon}{\theta_\varepsilon^2}\boldsymbol{\theta}_\varepsilon\times\left(\boldsymbol{\theta}_\varepsilon\times\boldsymbol{\omega}_\varepsilon\right). \tag{110}$$

The perturbed quantities $\boldsymbol{\omega}_\varepsilon$ and $\boldsymbol{\theta}_\varepsilon$ can be represented in the form

$$\boldsymbol{\omega}_\varepsilon = \boldsymbol{\omega} + \varepsilon\boldsymbol{\eta}, \qquad \boldsymbol{\theta}_\varepsilon = \boldsymbol{\theta} + \varepsilon\boldsymbol{\varphi}, \qquad |\varepsilon| \ll 1, \tag{111}$$

where $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ are defined by expressions (105)–(108). The quantities $\boldsymbol{\eta}$ and $\boldsymbol{\varphi}$ are called the first variations of $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ respectively. If we shall use representation (111), then we get the equations for $\boldsymbol{\eta}$ and $\boldsymbol{\varphi}$ with the varying coefficients. Because of this it will be better to represent the functions $\boldsymbol{\omega}_\varepsilon$ and $\boldsymbol{\theta}_\varepsilon$ in the next form

$$\boldsymbol{\omega}_\varepsilon = \mathbf{Q}(\psi\mathbf{k})\cdot\left(\boldsymbol{\omega}_0 + \varepsilon\boldsymbol{\eta}\right), \qquad \boldsymbol{\theta}_\varepsilon = \mathbf{Q}(\psi\mathbf{k})\cdot\left(\vartheta\mathbf{p} + \varepsilon\boldsymbol{\varphi}\right), \tag{112}$$

where the function ψ is defined by (108).

It is easy to calculate

$$\dot{\boldsymbol{\omega}}_\varepsilon = \mathbf{Q}(\psi\mathbf{k}) \cdot \left[\dot{\psi}\mathbf{k} \times \boldsymbol{\omega}_0 + \varepsilon\left(\dot{\boldsymbol{\eta}} + \dot{\psi}\mathbf{k} \times \boldsymbol{\eta}\right)\right],$$

$$\dot{\boldsymbol{\theta}}_\varepsilon = \mathbf{Q}(\psi\mathbf{k}) \cdot \left[\dot{\psi}\vartheta\mathbf{k} \times \mathbf{p} + \varepsilon\left(\dot{\boldsymbol{\varphi}} + \dot{\psi}\mathbf{k} \times \boldsymbol{\varphi}\right)\right].$$

From equations (110) the next equations for variations $\boldsymbol{\eta}$ and $\boldsymbol{\varphi}$ can be derived

$$A\left(\dot{\boldsymbol{\eta}} + \dot{\psi}\mathbf{k} \times \boldsymbol{\eta}\right) + c\boldsymbol{\varphi} = 0,$$

$$\dot{\boldsymbol{\varphi}} + \dot{\psi}\mathbf{k} \times \boldsymbol{\varphi} = \frac{\vartheta\sin\vartheta}{2(1-\cos\vartheta)}\boldsymbol{\eta} - \frac{\vartheta - \sin\vartheta}{2(1-\cos\vartheta)}(\mathbf{p}\cdot\boldsymbol{\varphi})\,\boldsymbol{\omega}_0 - \frac{1}{2}\boldsymbol{\varphi} \times \boldsymbol{\omega}_0 -$$
$$-\frac{1}{2}\vartheta\mathbf{p} \times \boldsymbol{\eta} + \frac{2(1-\cos\vartheta) - \vartheta\sin\vartheta}{2\vartheta(1-\cos\vartheta)}(\boldsymbol{\varphi}\cdot\boldsymbol{\omega}_0 + \vartheta\mathbf{p} \times \boldsymbol{\eta})\,\mathbf{p},$$

where $\dot{\psi}$ is determined by (108) and $\vartheta = \text{const}$. This system of linear differential equations with constant coefficients can be investigated by conventional methods. Our aim was only to show the derivation of the equations in variations.

Appendix 1. Elastic energy of foundation

In the section 3 there was given the definition of an elastic energy in terms of potential function $U(\boldsymbol{\theta})$. This function was interpreted as the elastic energy of foundation. However in the nonlinear theory of elasticity the concept of elastic energy has a uniquely determined meaning. Thus it is necessary to show that there is no contradiction between these two concepts.

The foundation is supposed to be an elastic body. The boundary of the foundation is the surface $S = S_1 \bigcup S_2 \bigcup S_3$. The part S_1 of the surface S is fixed. The part S_2 is a free surface. The part S_3 is the interface between the foundation and the rigid body.

Let us write the equation of the energy balance for the system "foundation plus rigid body"

$$\dot{K} + \dot{U} = 0, \tag{113}$$

where K is the kinetic energy of rigid body, since the foundation is supposed to be inertialess; U is the total intrinsic energy, i.e. elastic energy or energy of deformation, of the elastic foundation, since the intrinsic energy of rigid body has a constant value. The right side of (113) is equal to zero because the power of external forces is absent.

Now let us write the equation of the energy balance for rigid body only. The external forces, acting on the body, are generating by the vector of stress acting on the part S_3 of the boundary. Thus one can write

$$\dot{K} = -\int \mathbf{N}(P) \cdot \boldsymbol{\tau}(P) \cdot \dot{\mathbf{R}}(P)\, dS(P), \qquad P \in S_3, \tag{114}$$

where $\mathbf{R}(P)$ is the vector of position of the point P of the surface S_3; the integration is going over the surface S_3; the vector $\mathbf{N}$ is the external unit normal to the surface S_3; the tensor $\boldsymbol{\tau}$ is the Cauchy stress tensor.

In according with the basic theorem of kinematics of rigid body we have

$$\mathbf{R}(P) = \mathbf{R}(Q) + \mathbf{P}(t) \cdot (\mathbf{r}(P) - \mathbf{r}(Q)), \tag{115}$$

where Q is the pole, $\mathbf{r}(P)$ and $\mathbf{r}(Q)$ are the vectors of position of points P and Q in the reference position. From equation (115) it follows

$$\mathbf{v}(P) = \mathbf{v}(Q) + \boldsymbol{\omega}(t) \times [\mathbf{R}(P) - \mathbf{R}(Q)]. \tag{116}$$

Substituting expression (116) into equation (114) one can get

$$\dot{K} = \mathbf{F} \cdot \mathbf{v}(Q) + \mathbf{M}_e \cdot \boldsymbol{\omega}, \tag{117}$$

where

$$\mathbf{F} = -\int \mathbf{N}(P) \cdot \boldsymbol{\tau}(P)\, dS(P),$$

$$\mathbf{M}_e = -\int [\mathbf{R}(P) - \mathbf{R}(Q)] \times \boldsymbol{\tau}(P) \cdot \mathbf{N}(P)\, dS(P).$$

Making use of (113) equation (117) can be rewritten in the form

$$\mathbf{F} \cdot \mathbf{v}(Q) + \mathbf{M}_e \cdot \boldsymbol{\omega} = -\dot{U}(\mathbf{R}(Q), \boldsymbol{\theta}), \tag{118}$$

where the vector $\boldsymbol{\theta}$ is the vector of turn of the rigid body and henceforth of the surface S_3. If the point Q is fixed, then we have definition (21) or (46). Thus the potential U in expression (46) is the elastic energy of foundation.

Appendix 2. A derivation of the representation for the integrating tensor

Calculating the trace from the both sides of the Poisson equation (9) one can obtain

$$(\mathrm{tr}\mathbf{Q})^{\cdot} = \mathrm{tr}(\boldsymbol{\omega} \times \mathbf{Q}) = -2\frac{\sin\theta}{\theta}\boldsymbol{\theta} \cdot \boldsymbol{\omega}, \qquad \mathrm{tr}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}.$$

Taking into account the equality

$$\mathrm{tr}\mathbf{Q} = 1 + 2\cos\theta$$

from the previous equation it is easy to derive

$$\theta\dot{\theta} = \boldsymbol{\theta} \cdot \dot{\boldsymbol{\theta}} = \boldsymbol{\theta} \cdot \boldsymbol{\omega}. \tag{119}$$

Multiplying equation (9) by the vector $\boldsymbol{\theta}$ one can get

$$\dot{\mathbf{Q}} \cdot \boldsymbol{\theta} = \boldsymbol{\omega} \times \boldsymbol{\theta} = -\mathbf{R} \cdot \boldsymbol{\omega}$$

Making use the identity

$$\dot{\mathbf{Q}} \cdot \boldsymbol{\theta} = (\mathbf{Q} \cdot \boldsymbol{\theta})^{\cdot} - \mathbf{Q} \cdot \dot{\boldsymbol{\theta}} = -(\mathbf{Q} - \mathbf{E}) \cdot \dot{\boldsymbol{\theta}}$$

and equation (3) the previous equation can be rewritten in the form

$$\left(\frac{\sin\theta}{\theta}\mathbf{R} + \frac{1-\cos\theta}{\theta^2}\mathbf{R}^2\right)\cdot\dot{\boldsymbol{\theta}} = \mathbf{R}\cdot\boldsymbol{\omega}.$$

A general solution of this equation has the form

$$\boldsymbol{\omega} = \lambda\boldsymbol{\theta} + \left(\frac{\sin\theta}{\theta}\mathbf{E} + \frac{1-\cos\theta}{\theta^2}\mathbf{R}\right)\cdot\dot{\boldsymbol{\theta}}, \tag{120}$$

where the scalar function λ must be found.

Multiplying equation (120) by the vector $\boldsymbol{\theta}$ and taking into account equality (119) we have

$$\lambda = \frac{\theta-\sin\theta}{\theta^3}\boldsymbol{\theta}\cdot\dot{\boldsymbol{\theta}}.$$

Equation (120) takes the form

$$\boldsymbol{\omega} = \left[\mathbf{E} + \frac{1-\cos\theta}{\theta^2}\mathbf{R} + \frac{\theta-\sin\theta}{\theta^3}\mathbf{R}^2\right]\cdot\dot{\boldsymbol{\theta}} = \mathbf{Z}^{-1}\cdot\dot{\boldsymbol{\theta}}. \tag{121}$$

Here we use the identity

$$\mathbf{R}^2 = \boldsymbol{\theta}\otimes\boldsymbol{\theta} - \theta^2\mathbf{E}.$$

Expression (121) gives to us representation (20). Thus we had found the tensor $\mathbf{Z}^{-1}$. In order to calculate the tensor $\mathbf{Z}$ we must take into account that the tensor $\mathbf{Z}$ is the isotropic tensor function of the tensor $\mathbf{R}$. This means that the next representation is valid

$$\mathbf{Z} = \alpha\mathbf{E} + \beta\mathbf{R} + \gamma\mathbf{R}^2, \qquad \mathbf{Z}\cdot\mathbf{Z}^{-1} = \mathbf{E}.$$

From this it follows

$$\alpha = 1, \qquad \beta = -\frac{1}{2}, \qquad \gamma = \frac{1-g}{\theta^2}, \qquad g = \frac{\theta\sin\theta}{2\,(1-\cos\theta)}.$$

That is expression (11).

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Ferromagnets and Kelvin's Medium: Basic Equations and Magnetoacoustic Resonance*

Abstract

The nonlinear constitutive equations of Kelvin's medium (polar medium consisting of rotating particles) are obtained. It is shown that they include the known constitutive equations of ferromagnetic insulators as a particular case. Another way of taking the couplings of magnetic and elastic subsystems into account is suggested. Wave processes are investigated from this point of view. All results are interpreted both in terms of mechanical medium and ferromagnets.

1 Introduction

There are a lot of papers devoted to theories of elastic polar media. The first investigation in this field was developed by E. Cosserat and F. Cosserat [1]. Each particle of such a medium is a small rigid body (a point body). In [2], [3], [4] the linear theory for infinitesimal turns and displacements is considered.

In this paper we obtain a general form for *nonlinear* constitutive equations for Cosserat medium. Then we consider a special case of this medium — Kelvin's medium. Kelvin's medium is an elastic polar medium consisting of rotating particles with axial symmetry (Fig. 1). These particles can oscillate and rotate in general ways. Point bodies of this medium contrary to Cosserat continuum may have large angular velocities; displacements and turns may be finite. The idea to consider such a continuum was suggested by Lord Kelvin: "Kelvin imagined a model of a quasi-rigid ether built from gyrostates. The problem was to find a system resisting only to deformations concerned with rotation" [5].

We suppose that internal forces in this continuum do not depend on the angles of own rotation of particles or angular velocities of own rotation. We obtain the constitutive equations of this medium using the law of balance of energy via a phenomenological method suggested in [6]. This method allows to get nonlinear constitutive equations for elastic polar medium with particles of general kind, i.e. for generalized Cosserat medium.

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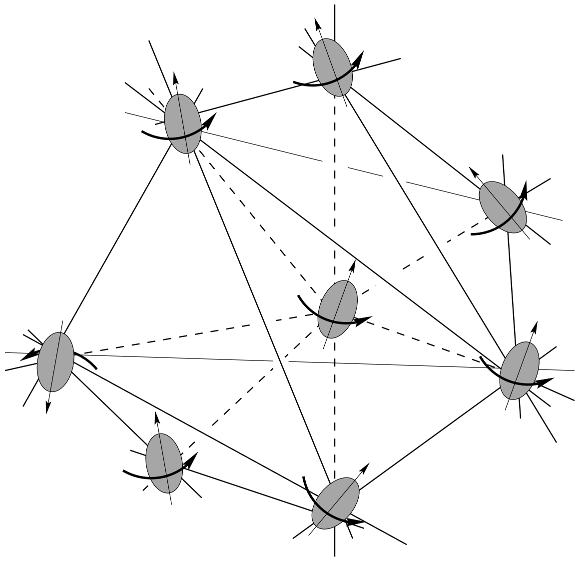

Figure 1: Kelvin's medium

We obtain these equations in section 2. Afterwards we take into account restrictions given by axial symmetry of particles and get constitutive equations of Kelvin's medium. We found these equations to be analogous to the constitutive equations of saturated elastic ferromagnetic insulators (see [7]) and to the constitutive equations in the non-classical theory of elastic shells [6]. There is also an exact analogy between dynamic equations of ferromagnets [7] and Kelvin's medium. This carries a similarity of wave processes in both media. We use the most general way of taking into account the coupling of translational and angular deformations in the function of strain energy. This allows us to describe phenomena analogous to magnetoacoustic resonance in ferromagnetic materials.

2 Dynamic and constitutive equations of Kelvin's medium

2.1 Kinematics of Kelvin's medium

We shall consider a deformable medium consisting of rotating particles with rotational symmetry having both translational and angular degrees of freedom.

Let q^s be material coordinates of a point of this medium, $\mathbf{r}(q^s)$ and $\mathbf{R}(q^s)$ are radius vectors of centre of mass of a point body in the initial and actual configuration respectively. [Here and further Roman subscripts take values 1,2,3 and Greek ones 1,2 and we shall employ the usual summation convention.] Let us associate with each point of this continuum an orthonormal vector basis $\mathbf{D}_k(q^s)$ that is "frozen" into a point body, where $\mathbf{m} \equiv \mathbf{D}_3$ is a unit vector of an axis of a point body. In the initial configuration let $\mathbf{D}_k = \mathbf{d}_k$, $\mathbf{m}_0 \equiv \mathbf{d}_3$. The dual basis $\mathbf{D}^k$ ($\mathbf{D}^k \cdot \mathbf{D}_i = \delta_i^k$) coincides with $\mathbf{D}_k$. We may introduce a turn-tensor $\mathbf{P} = \mathbf{D}_k \otimes \mathbf{d}^k$ that describes the turn of a point body. One can see that $\mathbf{D}_k = \mathbf{P} \cdot \mathbf{d}_k$. It is easy to show that $\mathbf{P} \cdot \mathbf{P}^\top = \mathbf{E}$, where $\mathbf{E}$ is a unit tensor, and that $\det \mathbf{P} = 1$. The turn-tensor can be represented in the form:

$$\mathbf{P}(t) = \mathbf{P}_3(\psi \mathbf{m}_0) \cdot \mathbf{P}_2(\vartheta \mathbf{l}_0) \cdot \mathbf{P}_1(\varphi \mathbf{m}_0), \tag{1}$$

where $\mathbf{P}_3(\psi \mathbf{m}_0) = (1 - \cos\psi)\mathbf{m}_0 \otimes \mathbf{m}_0 + \cos\psi \mathbf{E} + \sin\psi \mathbf{m}_0 \times \mathbf{E}$ is a turn-tensor about an axis $\mathbf{m}_0$ about an angle ψ etc., $\mathbf{l}_0$ and $\mathbf{m}_0$ are orthonormal vectors, ψ, ϑ, φ are angles of precession, nutation and own rotation respectively. We see that
$\mathbf{m}_0 \cdot \mathbf{P}_3 = \mathbf{m}_0 = \mathbf{P}_3 \cdot \mathbf{m}_0$ etc.

Let us denote $\mathbf{r}_i = \dfrac{\partial \mathbf{r}}{\partial q^i} \equiv \partial_i \mathbf{r}$ and $\mathbf{R}_i = \dfrac{\partial \mathbf{R}}{\partial q^i} \equiv \partial_i \mathbf{R}$. Nabla operators in the initial and actual configuration are defined by $\overset{\circ}{\nabla} = \mathbf{r}^i \partial_i$ and $\nabla = \mathbf{R}^i \partial_i$ respectively, where $\mathbf{r}^i$ and $\mathbf{R}^i$ are corresponding dual bases. We suppose $\overset{\circ}{\nabla} \mathbf{m}_0 = \mathbf{0}$ and put $\mathbf{r}^3 = \mathbf{m}_0$.

Let us introduce the following notation:

$\mathbf{u} = \mathbf{R} - \mathbf{r}$ is the translational displacement of a centre of mass of a point body;

$\mathbf{v} = \dot{\mathbf{R}}$ is the velocity of a centre of mass of a point body;

$\boldsymbol{\omega}(\mathbf{R}, t)$ is the angular velocity of a point body;

it can be defined by Poisson equation

$$\dot{\mathbf{P}} = \boldsymbol{\omega} \times \mathbf{P}; \tag{2}$$

and can be calculated as

$$\boldsymbol{\omega} = -[\dot{\mathbf{P}} \cdot \mathbf{P}^\top]_\times / 2; \tag{3}$$

or as

$$\boldsymbol{\omega} = \dot{\psi} \mathbf{m}_0 + \dot{\vartheta} \mathbf{P}_3 \cdot \mathbf{l}_0 + \dot{\varphi} \mathbf{P}_3 \cdot \mathbf{P}_2 \cdot \mathbf{m}_0 = \dot{\psi} \mathbf{m}_0 + \dot{\vartheta} \mathbf{l} + \dot{\varphi} \mathbf{m}, \tag{4}$$

where $\mathbf{l} = \mathbf{P}_3 \cdot \mathbf{l}_0$. One can see that $\dot{\mathbf{P}}_3 = \dot{\psi} \mathbf{m}_0 \times \mathbf{P}_3$ etc., and that
$\mathbf{l} \cdot \mathbf{m} = \mathbf{l}_0 \cdot \mathbf{P}_3^\top \cdot \mathbf{P}_3 \cdot \mathbf{P}_2 \cdot \mathbf{m}_0 = \mathbf{l}_0 \cdot \mathbf{m}_0 = 0$.

Let ut write an analog of Poisson equation for coordinate q^i instead of time t:

$$\partial_i \mathbf{P} = \boldsymbol{\Phi}_i \times \mathbf{P}, \tag{5}$$

here $\boldsymbol{\Phi}_i$ can be found as

$$\boldsymbol{\Phi}_i = -[\partial_i \mathbf{P} \cdot \mathbf{P}^\top]_\times / 2; \tag{6}$$

or as

$$\boldsymbol{\Phi}_i = \partial_i \psi \mathbf{m}_0 + \partial_i \vartheta \mathbf{P}_3 \cdot \mathbf{l}_0 + \partial_i \varphi \mathbf{P}_3 \cdot \mathbf{P}_2 \cdot \mathbf{m}_0 = \partial_i \psi \mathbf{m}_0 + \partial_i \vartheta \mathbf{l} + \partial_i \varphi \mathbf{m}. \tag{7}$$

Further we shall use relation

$$\partial_i \boldsymbol{\omega} = \dot{\boldsymbol{\Phi}}_i + \boldsymbol{\Phi}_i \times \boldsymbol{\omega} \tag{8}$$

Proof:

$$\begin{aligned}\partial_i\boldsymbol{\omega} = -\partial_i[\dot{\mathbf{P}}\cdot\mathbf{P}^\top]_\times/2 = -[(\partial_i\dot{\mathbf{P}})\cdot\mathbf{P}^\top + \dot{\mathbf{P}}\cdot\partial_i\mathbf{P}^\top]_\times/2 = \\ = -[(\boldsymbol{\Phi}_i\times\mathbf{P})^{\cdot}\cdot\mathbf{P}^\top + \boldsymbol{\omega}\times\mathbf{P}\cdot(\boldsymbol{\Phi}_i\times\mathbf{P})^\top]_\times/2 = \\ = -[\dot{\boldsymbol{\Phi}}_i\times\mathbf{E} + \boldsymbol{\Phi}_i\times(\boldsymbol{\omega}\times\mathbf{P})\cdot\mathbf{P}^\top - \boldsymbol{\omega}\times\mathbf{P}\cdot\mathbf{P}^\top\times\boldsymbol{\Phi}_i]_\times/2 = \\ = -[\dot{\boldsymbol{\Phi}}_i\times\mathbf{E} + \boldsymbol{\Phi}_i\times(\boldsymbol{\omega}\times\mathbf{E}) - \boldsymbol{\omega}\times\mathbf{E}\times\boldsymbol{\Phi}_i]_\times/2 = \dot{\boldsymbol{\Phi}}_i + \boldsymbol{\Phi}_i\times\boldsymbol{\omega}. \quad (9)\end{aligned}$$

Here we used (3) and (6).

Let us introduce strain tensors

$$\begin{aligned}\mathbf{A} &= \mathring{\nabla}\mathbf{R}\cdot\mathbf{P}, \\ \mathbf{K} &= \mathbf{r}^i\otimes\boldsymbol{\Phi}_i\cdot\mathbf{P}.\end{aligned} \quad (10)$$

Tensor $\mathbf{A}$ is responsible both for translational and angular deformation, and $\mathbf{K}$ is determined only by angular strain; using (7) we may get

$$\mathbf{K} = \mathring{\nabla}\psi\mathbf{m}_0\cdot\mathbf{P} + \mathring{\nabla}\vartheta\mathbf{l}\cdot\mathbf{P} + \mathring{\nabla}\varphi\mathbf{m}_0; \quad (11)$$

one can show that $\mathbf{K} = -(\mathring{\nabla}\mathbf{P}\cdot\mathbf{P}^\top)\cdot\cdot(\mathbf{E}\times\mathbf{P})/2$.

The density of kinetic energy is defined by

$$\mathcal{K} = \frac{1}{2}(\mathbf{v}\cdot\mathbf{v} + \boldsymbol{\omega}\cdot\boldsymbol{\Theta}\cdot\boldsymbol{\omega}), \quad (12)$$

where $\boldsymbol{\Theta} = \mathbf{P}\cdot\boldsymbol{\Theta}_0\cdot\mathbf{P}^\top$ is the density of central inertia tensor of a point body in the actual configuration, $\boldsymbol{\Theta}_0$ is the density of central inertia tensor of a point body in the initial configuration; since point bodies have an axial symmetry,

$$\boldsymbol{\Theta} = \lambda\mathbf{m}\otimes\mathbf{m} + \mu(\mathbf{E} - \mathbf{m}\otimes\mathbf{m}). \quad (13)$$

The density of impulse of the medium is given by the formula

$$\mathcal{K}_1 = \frac{\partial\mathcal{K}}{\partial\mathbf{v}} = \mathbf{v}, \quad (14)$$

and the density of kinetic moment calculated relatively to the origin is

$$\mathcal{K}_2 = \frac{\partial\mathcal{K}}{\partial\boldsymbol{\omega}} + \mathbf{R}\times\mathcal{K}_1 = \boldsymbol{\Theta}\cdot\boldsymbol{\omega} + \mathbf{R}\times\mathbf{v}. \quad (15)$$

2.2 Stress and couple tensors. Euler's laws of dynamics.

We denote

$\rho(\mathbf{R}, t)$ is the mass density in the actual configuration;

$\boldsymbol{\tau}(\mathbf{R}, t)$ is Cauchy stress-tensor; $\boldsymbol{\tau}_{(\mathbf{n})} = \mathbf{n}\cdot\boldsymbol{\tau}$, where $\boldsymbol{\tau}_{(\mathbf{n})}$ is stress vector acting upon the elementary surface, $\mathbf{n}$ is the normal to this surface;

$\boldsymbol{\mu}(\mathbf{R}, t)$ is the couple tensor which can be introduced analogously to the stress-tensor; $\boldsymbol{\mu}_{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\mu}$, where $\boldsymbol{\mu}_{(\mathbf{n})}$ is the moment acting upon the elementary surface with the normal $\mathbf{n}$;

$\mathbf{Q}(\mathbf{R}, t)$ is the density of the external force;

$\mathbf{L}(\mathbf{R}, t)$ is the density of the external moment;

$\mathcal{U}(\mathbf{R}, t)$ is the density of the strain energy.

Euler's first law of dynamics (balance of force) for a part of continuum $\triangle V$ bounded by a surfase Σ is

$$\frac{d}{dt} \int\limits_{\triangle V} \rho\, \mathcal{K}_1 \, dV = \int\limits_{\triangle V} \rho\, \mathbf{Q} \, dV + \int\limits_{\Sigma} \boldsymbol{\tau}_{(\mathbf{n})} \, d\Sigma. \tag{16}$$

It can be rewritten in a local form

$$\nabla \cdot \boldsymbol{\tau} + \rho\, \mathbf{Q} = \rho\, \ddot{\mathbf{u}}. \tag{17}$$

Euler's second law of dynamics (balance of moment) for a part of continuum $\triangle V$ bounded by a surfase Σ is

$$\frac{d}{dt} \int\limits_{\triangle V} \rho\, \mathcal{K}_2 \, dV = \int\limits_{\triangle V} \rho\, (\mathbf{L} + \mathbf{R} \times \mathbf{Q}) \, dV + \int\limits_{\Sigma} (\boldsymbol{\mu}_{(\mathbf{n})} + \mathbf{R} \times \boldsymbol{\tau}_{(\mathbf{n})}) \, d\Sigma. \tag{18}$$

One can rewrite it in a local form using Euler's first law of dynamics (16)

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_{\times} + \rho\, \mathbf{L} = \rho\, (\boldsymbol{\Theta} \cdot \boldsymbol{\omega})^{\cdot}. \tag{19}$$

If we consider the case when the densities of moments of inertia λ and μ are infinitesimal but the angular velocity of own rotation $\dot{\varphi}$ is large so that $\lambda\dot{\varphi} = O(1)$, (19) can be rewritten as

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_{\times} + \rho\, \mathbf{L} = \rho\, \boldsymbol{\omega} \times (\boldsymbol{\Theta} \cdot \boldsymbol{\omega}) + o\,(1). \tag{20}$$

Proof:

$$(\boldsymbol{\Theta} \cdot \boldsymbol{\omega})^{\cdot} = (\mathbf{P} \cdot \boldsymbol{\Theta}_0 \cdot \mathbf{P}^{\top} \cdot \boldsymbol{\omega})^{\cdot} \overset{(2)}{=} (\boldsymbol{\omega} \times \mathbf{P} \cdot \boldsymbol{\Theta}_0 \cdot \mathbf{P}^{\top} - \mathbf{P} \cdot \boldsymbol{\Theta}_0 \cdot \mathbf{P}^{\top} \times \boldsymbol{\omega}) \cdot \boldsymbol{\omega} + \\ + \mathbf{P} \cdot \boldsymbol{\Theta}_0 \cdot \mathbf{P}^{\top} \cdot \dot{\boldsymbol{\omega}} = \boldsymbol{\omega} \times (\boldsymbol{\Theta} \cdot \boldsymbol{\omega}) + \boldsymbol{\Theta} \cdot \dot{\boldsymbol{\omega}} \tag{21}$$

Since we suppose all interactions to be potential, there can not exist any internal stresses that induce own rotation of a point body, and $\ddot{\varphi}$ is of the same order as $\mathbf{L}$. Taking this in account we may write

$$\boldsymbol{\Theta} \cdot \dot{\boldsymbol{\omega}} \overset{(4)}{=} \boldsymbol{\Theta} \cdot (\ddot{\psi}\mathbf{m}_0 + \ddot{\vartheta}\mathbf{l} + \ddot{\varphi}\mathbf{m} + \dot{\vartheta}\dot{\psi}\mathbf{m}_0 \times \mathbf{l} + \dot{\varphi}\, \boldsymbol{\omega} \times \mathbf{m}) \overset{(13)}{=} \\ = (\lambda\mathbf{m} \otimes \mathbf{m} + \mu(\mathbf{E} - \mathbf{m} \otimes \mathbf{m})) \cdot (\ddot{\psi}\mathbf{m}_0 + \ddot{\vartheta}\mathbf{l} + \ddot{\varphi}\mathbf{m} + \dot{\vartheta}\dot{\psi}\mathbf{m}_0 \times \mathbf{l} + \dot{\varphi}\, \boldsymbol{\omega} \times \mathbf{m}) = \\ = \mu\dot{\varphi}\, \boldsymbol{\omega} \times \mathbf{m} + o\,(1) \overset{(4)}{=} \mu\dot{\varphi}(\dot{\psi}\mathbf{m}_0 \times \mathbf{m} + \dot{\vartheta}\mathbf{l} \times \mathbf{m}) + o\,(1) = \\ = O(1)(\dot{\psi}\mathbf{m}_0 \times \mathbf{m} + \dot{\vartheta}\mathbf{l} \times \mathbf{m}) + o\,(1) = o\,(1) \tag{22}$$

It is easy to show that under these conditions $\boldsymbol{\omega} \times \boldsymbol{\Theta} \cdot \boldsymbol{\omega} = O(1)$. Indeed,
$\boldsymbol{\omega} \times \boldsymbol{\Theta} \cdot \boldsymbol{\omega} \overset{(13)}{=} \boldsymbol{\omega} \times (\lambda - \mu)\mathbf{m} \otimes \mathbf{m} \cdot \boldsymbol{\omega} + \mu \boldsymbol{\omega} \times \boldsymbol{\omega} \overset{(4)}{=} (\lambda - \mu)(\dot{\varphi} + \dot{\psi}\mathbf{m}_0 \cdot \mathbf{m})(\dot{\psi}\mathbf{m}_0 \times \mathbf{m} + \dot{\vartheta}\mathbf{l} \times \mathbf{m})$

which is $O(1)$ in general case since $\lambda\dot{\varphi} = O(1)$.

Thus we have $(\boldsymbol{\Theta} \cdot \boldsymbol{\omega})^{\cdot} = \boldsymbol{\omega} \times (\boldsymbol{\Theta} \cdot \boldsymbol{\omega}) + o(1)$ and (19) may be rewritten as (20) provided $\lambda = o(1)$, $\mu = o(1)$, $\lambda\dot{\varphi} = O(1)$, $\dot{\psi} = O(1)$, $\dot{\vartheta} = O(1)$.

NB: Under these conditions

$$\boldsymbol{\Theta} \cdot \boldsymbol{\omega} = \lambda\dot{\varphi}\mathbf{m} + \boldsymbol{\Theta} \cdot (\dot{\psi}\mathbf{m}_0 + \dot{\vartheta}\mathbf{l}) = \lambda\dot{\varphi}\mathbf{m} + o(1). \tag{23}$$

2.3 Nonlinear constitutive equations

2.3.1 Nonlinear constitutive equations for generalized Cosserat medium

We obtain the nonlinear constitutive equations for elastic polar medium via the method used in the theory of shells (P.A. Zhilin, [6]).

The equation for balance of energy for a polar medium is:

$$\frac{d}{dt}\int\limits_{\triangle V} \rho(\mathcal{K} + \mathcal{U})\, dV = \int\limits_{\triangle V} \rho(\mathbf{Q} \cdot \mathbf{v} + \mathbf{L} \cdot \boldsymbol{\omega})\, dV + \int\limits_{\Sigma} (\boldsymbol{\tau}_{(\mathbf{n})} \cdot \mathbf{v} + \boldsymbol{\mu}_{(\mathbf{n})} \cdot \boldsymbol{\omega})\, d\Sigma. \tag{24}$$

Its local form is

$$\rho\dot{\mathcal{U}} = \boldsymbol{\tau}^{\top} \cdot\cdot \nabla\mathbf{v} - \boldsymbol{\tau}_{\times} \cdot \boldsymbol{\omega} + \boldsymbol{\mu}^{\top} \cdot\cdot \nabla\boldsymbol{\omega}. \tag{25}$$

It can be rewritten in the form

$$\rho\dot{\mathcal{U}} = \boldsymbol{\tau}_*^{\top} \cdot\cdot \dot{\mathbf{A}} + \boldsymbol{\mu}_*^{\top} \cdot\cdot \dot{\mathbf{K}}, \tag{26}$$

where $\boldsymbol{\tau}_* = \overset{\circ}{\nabla}\mathbf{R}^{-\top} \cdot \boldsymbol{\tau} \cdot \mathbf{P}$ is the energetical stress tensor, $\boldsymbol{\mu}_* = \overset{\circ}{\nabla}\mathbf{R}^{-\top} \cdot \boldsymbol{\mu} \cdot \mathbf{P}$ is the energetical couple tensor.
Proof:

$$\begin{aligned}
\boldsymbol{\tau}_*^{\top} \cdot\cdot \dot{\mathbf{A}} + \boldsymbol{\mu}_*^{\top} \cdot\cdot \dot{\mathbf{K}} &\overset{(10)}{=} (\mathbf{P}^{\top} \cdot \boldsymbol{\tau}^{\top} \cdot (\overset{\circ}{\nabla}\mathbf{R})^{-1}) \cdot\cdot (\overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{P})^{\cdot} + (\mathbf{P}^{\top} \cdot \boldsymbol{\mu}^{\top} \cdot (\overset{\circ}{\nabla}\mathbf{R})^{-1}) \cdot\cdot (\mathbf{r}^i \otimes \boldsymbol{\Phi}_i \cdot \mathbf{P})^{\cdot} \overset{(2)}{=} \\
&= (\mathbf{P}^{\top} \cdot \boldsymbol{\tau}^{\top} \cdot (\overset{\circ}{\nabla}\mathbf{R})^{-1}) \cdot\cdot (\overset{\circ}{\nabla}\mathbf{v} \cdot \mathbf{P}) + (\mathbf{P}^{\top} \cdot \boldsymbol{\tau}^{\top} \cdot (\overset{\circ}{\nabla}\mathbf{R})^{-1}) \cdot\cdot (\overset{\circ}{\nabla}\mathbf{R} \cdot (\boldsymbol{\omega} \times \mathbf{P})) + \\
&+ (\mathbf{P}^{\top} \cdot \boldsymbol{\mu}^{\top} \cdot (\overset{\circ}{\nabla}\mathbf{R})^{-1}) \cdot\cdot (\mathbf{r}^i \otimes \dot{\boldsymbol{\Phi}}_i \cdot \mathbf{P}) + (\mathbf{P}^{\top} \cdot \boldsymbol{\mu}^{\top} \cdot (\overset{\circ}{\nabla}\mathbf{R})^{-1}) \cdot\cdot (\mathbf{r}^i \otimes \boldsymbol{\Phi}_i \cdot (\boldsymbol{\omega} \times \mathbf{P})) = \\
&= \boldsymbol{\tau}^{\top} \cdot\cdot ((\overset{\circ}{\nabla}\mathbf{R})^{-1} \cdot \overset{\circ}{\nabla}\mathbf{v}) + \boldsymbol{\tau}^{\top} \cdot\cdot ((\overset{\circ}{\nabla}\mathbf{R})^{-1} \cdot \overset{\circ}{\nabla}\mathbf{R} \cdot (\boldsymbol{\omega} \times \mathbf{E})) + \boldsymbol{\mu}^{\top} \cdot\cdot ((\overset{\circ}{\nabla}\mathbf{R})^{-1} \cdot \mathbf{r}^i \otimes \dot{\boldsymbol{\Phi}}_i) + \\
&+ \boldsymbol{\mu}^{\top} \cdot\cdot ((\overset{\circ}{\nabla}\mathbf{R})^{-1} \cdot \mathbf{r}^i \otimes \dot{\boldsymbol{\Phi}}_i \cdot (\boldsymbol{\omega} \times \mathbf{E})) = \boldsymbol{\tau}^{\top} \cdot\cdot \nabla\mathbf{v} + \boldsymbol{\tau}^{\top} \cdot\cdot (\boldsymbol{\omega} \times \mathbf{E}) + \boldsymbol{\mu}^{\top} \cdot\cdot \mathbf{R}^i \otimes (\dot{\boldsymbol{\Phi}}_i + \boldsymbol{\Phi}_i \times \boldsymbol{\omega}) \overset{(8)}{=} \\
&= \boldsymbol{\tau}^{\top} \cdot\cdot \nabla\mathbf{v} - \boldsymbol{\tau}_{\times} \cdot \boldsymbol{\omega} + \boldsymbol{\mu}^{\top} \cdot\cdot \nabla\boldsymbol{\omega}
\end{aligned} \tag{27}$$

We define elastic medium as a medium where the density of strain energy depends only on the deformation, i.e. $\mathcal{U} = \mathcal{U}(\mathbf{A}, \mathbf{K})$.

Formula (26) allows us to get correct nonlinear constitutive equations of an elastic polar medium [6]:

$$\boldsymbol{\tau} = \rho\, \overset{\circ}{\nabla}\mathbf{R}^\top \cdot \frac{\partial \mathcal{U}}{\partial \mathbf{A}} \cdot \mathbf{P}^\top, \tag{28}$$

$$\boldsymbol{\mu} = \rho\, \overset{\circ}{\nabla}\mathbf{R}^\top \cdot \frac{\partial \mathcal{U}}{\partial \mathbf{K}} \cdot \mathbf{P}^\top, \tag{29}$$

where $\mathcal{U} = \mathcal{U}(\mathbf{A}, \mathbf{K})$.

2.3.2 Nonlinear constitutive equations for Kelvin's medium

Now let us take into account that Kelvin's medium is a medium of a special kind. In this case strain energy $\mathcal{U}$ is not a function of general kind in $\mathbf{A}$ and $\mathbf{K}$, since we assume that $\mathcal{U}(\mathbf{R})$ does not depend on φ or $\nabla\varphi$. Let us search for functions in $\mathbf{A}$ and $\mathbf{K}$ (strain tensors) such that we satisfy these restrictions whenever $\mathcal{U}$ depends only on these tensors. We shall use mathematical methods that one can find in [9].

The equation

$$\frac{\partial \mathcal{U}}{\partial \varphi} = 0 \tag{30}$$

can be rewritten as

$$\left(\frac{\partial \mathcal{U}}{\partial \mathbf{A}}\right)^\top \cdot\cdot (\mathbf{A} \times \mathbf{m}_0) + \left(\frac{\partial \mathcal{U}}{\partial \mathbf{K}}\right)^\top \cdot\cdot (\mathbf{K} \times \mathbf{m}_0) = 0. \tag{31}$$

Characteristic equations for (31) are

$$\frac{\partial \mathbf{A}}{\partial \varphi} = \mathbf{A} \times \mathbf{m}_0, \qquad \frac{\partial \mathbf{K}}{\partial \varphi} = \mathbf{K} \times \mathbf{m}_0. \tag{32}$$

Density of strain energy $\mathcal{U}$ is a function of first integrals of (32). These integrals are strain tensors for medium under consideration. In the shell theory (P.A. Zhilin, [6]), the above system of equations occurs because own rotation of a shell fibre must not influence the energy of deformation. In case of a shell, the system has order 12 because we consider a shell to be a 2D object. In case of a 3D continuum, the system has order 18.

There are various possibilities in choosing the set of first integrals of (32), i.e. strain tensors of Kelvin's medium:

1. This set of functions includes all first integrals of (32):

$$\begin{aligned}
&\boldsymbol{\mathcal{E}} = (\mathbf{A} \cdot \mathbf{A}^\top - \mathbf{E})/2 = (\overset{\circ}{\nabla}\mathbf{R} \cdot \overset{\circ}{\nabla}\mathbf{R}^\top - \mathbf{E})/2,\\
&\mathbf{F} = \mathbf{K} \cdot \tilde{\mathbf{a}} \cdot \mathbf{A}^\top = (\overset{\circ}{\nabla}\psi \otimes \mathbf{m}_0 + \overset{\circ}{\nabla}\vartheta \otimes \mathbf{l}) \cdot \mathbf{P} \cdot \tilde{\mathbf{a}} \cdot \mathbf{P}^\top \cdot \overset{\circ}{\nabla}\mathbf{R}^\top,\\
&\boldsymbol{\gamma} = \mathbf{A} \cdot \mathbf{m}_0 = \overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{m},\\
&\boldsymbol{\xi} = \mathbf{K} \cdot \mathbf{m}_0 \overset{(11)}{=} \overset{\circ}{\nabla}\psi \cos\vartheta + \overset{\circ}{\nabla}\varphi,
\end{aligned} \tag{33}$$

where $\mathbf{l} = \mathbf{P} \cdot \mathbf{l}_0$ and we can choose either $\tilde{\mathbf{a}} = \mathbf{E} - \mathbf{m}_0 \otimes \mathbf{m}_0$ or $\tilde{\mathbf{a}} = \mathbf{E} \times \mathbf{m}_0$. We see that $\boldsymbol{\mathcal{E}}$ is Cauchy–Green strain tensor. Tensor $\mathbf{F}$ corresponds to the "mixed" translational-angular strain.

We see that in (33) only vector $\boldsymbol{\xi}$ depends on $\nabla\varphi$. Thus we conclude that $\mathcal{U}$ does not depend on $\boldsymbol{\xi}$.

From (28), (29) we get the corresponding constitutive equations:

$$
\begin{aligned}
&\boldsymbol{\tau}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\left(\frac{\partial\mathcal{U}}{\partial\boldsymbol{\mathcal{E}}}\cdot\mathbf{A}+\frac{\partial\mathcal{U}}{\partial\boldsymbol{\gamma}}\otimes\mathbf{m}_0+\left(\frac{\partial\mathcal{U}}{\partial\mathbf{F}}\right)^{\top}\cdot\mathbf{K}\cdot\tilde{\mathbf{a}}^{\top}\right)\cdot\mathbf{P}^{\top},\\
&\boldsymbol{\mu}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\,\frac{\partial\mathcal{U}}{\partial\mathbf{F}}\cdot\mathbf{A}\cdot\tilde{\mathbf{a}}^{\top}\cdot\mathbf{P}^{\top},\\
&\mathcal{U}=\mathcal{U}(\boldsymbol{\mathcal{E}},\mathbf{F},\boldsymbol{\gamma}).
\end{aligned}
\tag{34}
$$

2. One can suggest another set of first integrals for (32):

$$
\begin{aligned}
&\boldsymbol{\Phi}=\mathbf{K}\cdot\mathbf{a}\cdot\mathbf{K}^{\top}=\sin^2\vartheta\,\overset{\circ}{\nabla}\psi\otimes\overset{\circ}{\nabla}\psi+\overset{\circ}{\nabla}\vartheta\otimes\overset{\circ}{\nabla}\vartheta,\\
&\boldsymbol{\mathcal{E}},\quad \alpha=\mathbf{m}_0\cdot\mathbf{F}\cdot\mathbf{m}_0,\quad \boldsymbol{\gamma},\quad \boldsymbol{\xi},
\end{aligned}
\tag{35}
$$

where $\mathbf{a}=\mathbf{E}-\mathbf{m}_0\otimes\mathbf{m}_0$. Here $\boldsymbol{\Phi}$ is responsible for angular deformations (like $\boldsymbol{\mathcal{E}}$ for translational strain), and α corresponds to the "mixed" kind of deformation.

Omitting $\boldsymbol{\xi}$ from this set for the reason mentioned above, we obtain the following constitutive equations:

$$
\begin{aligned}
&\boldsymbol{\tau}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\left(\frac{\partial\mathcal{U}}{\partial\boldsymbol{\mathcal{E}}}\cdot\mathbf{A}+\frac{\partial\mathcal{U}}{\partial\boldsymbol{\gamma}}\otimes\mathbf{m}_0+\frac{\partial\mathcal{U}}{\partial\alpha}\,\mathbf{m}_0\otimes\mathbf{m}_0\cdot\mathbf{K}\cdot\mathbf{a}\right)\cdot\mathbf{P}^{\top},\\
&\boldsymbol{\mu}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\left(2\frac{\partial\mathcal{U}}{\partial\boldsymbol{\Phi}}\cdot\mathbf{K}\cdot\mathbf{a}+\frac{\partial\mathcal{U}}{\partial\alpha}\,\mathbf{m}_0\otimes\mathbf{m}_0\cdot\mathbf{A}\cdot\mathbf{a}\right)\cdot\mathbf{P}^{\top},\\
&\mathcal{U}=\mathcal{U}(\boldsymbol{\mathcal{E}},\boldsymbol{\Phi},\boldsymbol{\gamma},\alpha).
\end{aligned}
\tag{36}
$$

Of course these two variants are not the only ones possible; in fact, there is an infinite amount of sets of first integrals of (32).

Set (33) is a set of independent integrals of (32) in the case of shells. In the case under consideration, (33) as well as (35) include all independent integrals and some dependent ones. There are many ways of eliminating dependent functions. For example, this is a set of independent integrals:

$$
\boldsymbol{\mathcal{E}}_1=\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{E}}\cdot\cdot\,\mathbf{m}_0\otimes\mathbf{m}_0\otimes\mathbf{m}_0\otimes\mathbf{m}_0,\quad \mathbf{F}\cdot\mathbf{a},\quad \boldsymbol{\gamma},\quad \boldsymbol{\xi}.
\tag{37}
$$

We also can consider another set of independent integrals:

$$
\boldsymbol{\mathcal{E}}_1,\quad \boldsymbol{\Phi}_1=\boldsymbol{\Phi}-\boldsymbol{\Phi}\cdot\cdot\,\mathbf{m}_0\otimes\mathbf{m}_0\otimes\mathbf{m}_0\otimes\mathbf{m}_0,\quad \alpha,\quad \boldsymbol{\gamma},\quad \boldsymbol{\xi}.
\tag{38}
$$

The density of strain energy $\mathcal{U}$ depends only on these functions. Excluding $\boldsymbol{\xi}$ from these sets we have $\mathcal{U}=\mathcal{U}(\boldsymbol{\mathcal{E}}_1,\mathbf{F}\cdot\mathbf{a},\boldsymbol{\gamma})$ or $\mathcal{U}=\mathcal{U}(\boldsymbol{\mathcal{E}}_1,\boldsymbol{\Phi}_1,\boldsymbol{\gamma},\alpha)$.

Corresponding to (37) and (38), the constitutive equations are:

$$
\begin{aligned}
&\boldsymbol{\tau}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\left(\frac{\partial\mathcal{U}}{\partial\boldsymbol{\mathcal{E}}_1}\cdot\mathbf{A}+\frac{\partial\mathcal{U}}{\partial\boldsymbol{\gamma}}\otimes\mathbf{m}_0+\left(\frac{\partial\mathcal{U}}{\partial\mathbf{F}\cdot\mathbf{a}}\right)^{\top}\cdot\mathbf{K}\cdot\tilde{\mathbf{a}}^{\top}\right)\cdot\mathbf{P}^{\top},\\
&\boldsymbol{\mu}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\,\frac{\partial\mathcal{U}}{\partial\mathbf{F}\cdot\mathbf{a}}\cdot\mathbf{a}\cdot\mathbf{A}\cdot\tilde{\mathbf{a}}^{\top}\cdot\mathbf{P}^{\top},\\
&\mathcal{U}=\mathcal{U}(\boldsymbol{\mathcal{E}}_1,\mathbf{F}\cdot\mathbf{a},\boldsymbol{\gamma})
\end{aligned}
\tag{39}
$$

and

$$
\begin{aligned}
&\boldsymbol{\tau}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\left(\frac{\partial\mathcal{U}}{\partial\boldsymbol{\mathcal{E}}_1}\cdot\mathbf{A}+\frac{\partial\mathcal{U}}{\partial\boldsymbol{\gamma}}\otimes\mathbf{m}_0+\frac{\partial\mathcal{U}}{\partial\alpha}\,\mathbf{m}_0\otimes\mathbf{m}_0\cdot\mathbf{K}\cdot\mathbf{a}\right)\cdot\mathbf{P}^{\top},\\
&\boldsymbol{\mu}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\left(2\frac{\partial\mathcal{U}}{\partial\boldsymbol{\Phi}_1}\cdot\mathbf{K}\cdot\mathbf{a}+\frac{\partial\mathcal{U}}{\partial\alpha}\,\mathbf{m}_0\otimes\mathbf{m}_0\cdot\mathbf{A}\cdot\mathbf{a}\right)\cdot\mathbf{P}^{\top},\\
&\mathcal{U}=\mathcal{U}(\boldsymbol{\mathcal{E}}_1,\boldsymbol{\Phi}_1,\boldsymbol{\gamma},\alpha).
\end{aligned}
\tag{40}
$$

respectively.

It is possible to use any of (33), (35), (37), (38) as strain tensors. If we use (33) or (35), twice or more do we take into account dependence $\mathcal{U}$ on certain kinds of strain. If we use (37) or (38) and consider the simplest nonlinear theory (taking $\mathcal{U}$ to be the quadratic form of the strain tensors), $\mathcal{U}$ will depend on chosen strain tensors (37) or (38) in a simple way, and on other (dependent) strain tensors in a complicated way. The latter seems to be less convenient.

NB: Set of the functions (33) as well as (35) includes all independent kinds of deformation that induce stresses in Kelvin's medium. If $\mathcal{U}$ depends of any other kind of deformation, the latter can be expressed as a function of strain tensors (33) or (35). If we omit α from (33), this set will not contain all independent strain tensors. Tensor $\mathbf{F}$ in (33) and α in (35) are special "mixed" kinds of deformation that depend on the product of the translational displacements gradient and the gradient of a turn-tensor of a point body. If $\mathcal{U}$ depends on this kind of deformation, this is sufficient for existence of a coupling between translational and angular displacements.

2.3.3 Restrictions on the stress and couple tensors for Kelvin's medium

The density of strain energy in Kelvin's medium has to satisfy restrictions $\dfrac{\partial\mathcal{U}}{\partial\varphi}=0$ and $\dfrac{\partial\mathcal{U}}{\partial\nabla\varphi}=0$. In the subsection above we have rewritten these restrictions in terms of strain energy and strain tensors. Now let us get another form for them in terms of stress tensors.

The fact that internal stresses in Kelvin's medium can not be induced by a gradient of own rotation of its particles having axial symmetry can be rewritten as $\boldsymbol{\mu}\cdot\mathbf{m}=\mathbf{0}$. Indeed, using (29) we may write

$$
\begin{aligned}
\boldsymbol{\mu}\cdot\mathbf{m}=\rho\,\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\frac{\partial\mathcal{U}}{\partial\mathbf{K}}\cdot\mathbf{P}^{\top}\cdot\mathbf{m}=\rho\,\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\frac{\partial\mathcal{U}}{\partial\mathbf{K}}\cdot\mathbf{m}_0=\\
=\rho\,\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\frac{\partial\mathcal{U}}{\partial\mathbf{K}\cdot\mathbf{m}_0}=\rho\,\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\frac{\partial\mathcal{U}}{\partial\boldsymbol{\xi}}=\mathbf{0},
\end{aligned}
\tag{41}
$$

since $\boldsymbol{\xi}=\overset{\circ}{\nabla}\psi\cos\theta+\overset{\circ}{\nabla}\varphi$. Thus we see that

$$
\frac{\partial\mathcal{U}}{\partial\nabla\varphi}=0 \quad\Longleftrightarrow\quad \boldsymbol{\mu}\cdot\mathbf{m}=\mathbf{0}. \tag{42}
$$

Our assumption $\dfrac{\partial\mathcal{U}}{\partial\varphi}=0$ involves the analog to the "6th balance equation" in the theory of shells [6]:

$$
\boldsymbol{\tau}_{\times}\cdot\mathbf{m}=\boldsymbol{\mu}^{\top}\cdot\cdot\nabla\mathbf{m}. \tag{43}
$$

This can be got from (34) or (36), but it is more easy to get it directly from (28), (29). Let us transform the left-hand side of (43):

$$\boldsymbol{\tau}_\times \cdot \mathbf{m} \overset{(28)}{=} -\rho\,(\mathbf{P}\cdot\left(\frac{\partial\mathcal{U}}{\partial\mathbf{A}}\right)^\top \cdot \overset{\circ}{\nabla}\mathbf{R})\cdot\cdot(\mathbf{E}\times\mathbf{m}) = -\rho\left(\frac{\partial\mathcal{U}}{\partial\mathbf{A}}\right)^\top \cdot\cdot((\overset{\circ}{\nabla}\mathbf{R}\times\mathbf{m})\cdot\mathbf{P}) =$$
$$= -\rho\left(\frac{\partial\mathcal{U}}{\partial\mathbf{A}}\right)^\top \cdot\cdot((\overset{\circ}{\nabla}\mathbf{R}\cdot\mathbf{P})\times(\mathbf{P}^\top\cdot\mathbf{m})) \overset{(10)}{=} -\rho\left(\frac{\partial\mathcal{U}}{\partial\mathbf{A}}\right)^\top \cdot\cdot(\mathbf{A}\times\mathbf{m}_0). \quad (44)$$

Now we shall transform the right-hand side of (43):

$$\boldsymbol{\mu}^\top\cdot\cdot\nabla\mathbf{m} \overset{(29)}{=} \rho\,(\mathbf{P}\cdot\left(\frac{\partial\mathcal{U}}{\partial\mathbf{K}}\right)^\top \cdot\overset{\circ}{\nabla}\mathbf{R})\cdot\cdot\nabla\mathbf{m} \overset{(5)}{=} \rho\,(\mathbf{P}\cdot\left(\frac{\partial\mathcal{U}}{\partial\mathbf{K}}\right)^\top \cdot\mathbf{r}^s\otimes\mathbf{R}_s)\cdot\cdot(\mathbf{R}^i\otimes\boldsymbol{\Phi}_i\times\mathbf{m}) =$$
$$= \rho\left(\frac{\partial\mathcal{U}}{\partial\mathbf{K}}\right)^\top \cdot\cdot(\mathbf{r}^i\otimes(\boldsymbol{\Phi}_i\times\mathbf{m})\cdot\mathbf{P}) = \rho\left(\frac{\partial\mathcal{U}}{\partial\mathbf{K}}\right)^\top \cdot\cdot(\mathbf{r}_i\otimes(\mathbf{P}^\top\cdot\boldsymbol{\Phi}_i)\times(\mathbf{P}^\top\cdot\mathbf{m})) =$$
$$= \rho\left(\frac{\partial\mathcal{U}}{\partial\mathbf{K}}\right)^\top \cdot\cdot((\mathbf{r}^i\otimes\boldsymbol{\Phi}_i\cdot\mathbf{P})\times\mathbf{m}_0) \overset{(10)}{=} \rho\left(\frac{\partial\mathcal{U}}{\partial\mathbf{K}}\right)^\top \cdot\cdot(\mathbf{K}\times\mathbf{m}_0). \quad (45)$$

Thus we have

$$\boldsymbol{\mu}^\top\cdot\cdot\nabla\mathbf{m} - \boldsymbol{\tau}_\times\cdot\mathbf{m} = \rho\left(\frac{\partial\mathcal{U}}{\partial\mathbf{K}}\right)^\top \cdot\cdot(\mathbf{K}\times\mathbf{m}_0) + \rho\left(\frac{\partial\mathcal{U}}{\partial\mathbf{A}}\right)^\top \cdot\cdot(\mathbf{A}\times\mathbf{m}_0) =$$
$$= \rho\left(\frac{\partial\mathcal{U}}{\partial\mathbf{K}}\right)^\top \cdot\cdot\frac{\partial\mathbf{K}}{\partial\varphi} + \rho\left(\frac{\partial\mathcal{U}}{\partial\mathbf{A}}\right)^\top \cdot\cdot\frac{\partial\mathbf{A}}{\partial\varphi} = \rho\,\frac{\partial\mathcal{U}}{\partial\varphi}, \quad (46)$$

and we may conclude that

$$\frac{\partial\mathcal{U}}{\partial\varphi} = 0 \quad \Longleftrightarrow \quad \boldsymbol{\tau}_\times\cdot\mathbf{m} = \boldsymbol{\mu}^\top\cdot\cdot\nabla\mathbf{m}. \quad (47)$$

At the same time in general case $\boldsymbol{\tau}_\times\cdot\mathbf{m} \neq 0$.

If $\mathcal{U} = \mathcal{U}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \boldsymbol{\gamma})$, i.e. $\mathcal{U}$ does not depend on α, formula (36) yields

$$\boldsymbol{\tau}_\times\cdot\mathbf{m} = \boldsymbol{\mu}^\top\cdot\cdot\nabla\mathbf{m} = 0, \quad (48)$$

but if we assume this, we lose the dependence on one of the kinds of deformations that can exist and is not forbidden by thermodynamics.

Later it will be shown that (48) is valid in the linear theory. It means that the linear theory can not take into account dependence of strain energy on all kinds of "mixed" deformation, i.e. describe completely an interaction between angular and translational subsystems.

NB: The angular velocity of own rotation $\dot{\varphi}$ can not be changed by any internal forces or moments since they do not perform mechanical work on own rotation of the body. If external body moment has no projection on the axis $\mathbf{m}$ of a point body, $\dot{\varphi}$ does not depend on time and can be considered a constant of the medium.

2.4 Linear constitutive equations

Let us assume that in the initial configuration stresses are equal to zero, and that $\overset{\circ}{\nabla}\mathbf{m}_0 = \mathbf{0}$. Let angles of nutation and translational displacements to be infinitesimal, i.e.

$$\mathbf{P} \approx (\mathbf{E} + \theta \times \mathbf{E}) \cdot \mathbf{P}_1(\varphi \mathbf{m}_0), \;\; \theta = o(1), \quad \mathbf{u} = \mathbf{R} - \mathbf{r} = o(1). \tag{49}$$

It is possible to obtain the linear theory by different ways. The simplest one is to expand the law of energy balance (25) and to require independence $\rho\dot{\mathcal{U}}$ on the angular velocity of own rotation $\dot{\varphi}$. After that one will obtain the linear analog for (26) and linearized restrictions (47), (42). It gives the possibility to get linear constitutive equations. It was done in [10].

We shall obtain the linear theory from the nonlinear one. We shall use notation $[\cdot]_n$ the term of order n in $\mathbf{u}, \theta$. One can see that $[\mathbf{A}]_0 = \mathbf{P}_1(\varphi\mathbf{m}_0), [\mathbf{K}]_0 = \mathbf{0}$, and

$$[\mathbf{A}]_1 = \mathbf{g} \cdot \mathbf{P}_1(\varphi\mathbf{m}_0), \;\; [\mathbf{K}]_1 = \mathbf{f} \cdot \mathbf{P}_1(\varphi\mathbf{m}_0), \;\; \mathbf{g} = \overset{\circ}{\nabla}\mathbf{u} + \theta \times \mathbf{E}, \;\; \mathbf{f} = \overset{\circ}{\nabla}\theta. \tag{50}$$

We shall expand nonlinear constitutive equations (34) with $\tilde{\mathbf{a}} = \mathbf{a}$.

$$[\boldsymbol{\tau}]_0 = \rho_0 \left(\left[\frac{\partial \mathcal{U}}{\partial \mathcal{E}}\right]_0 + \left[\frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}}\right]_0 \otimes \mathbf{m}_0 \right), \tag{51}$$

$$[\boldsymbol{\mu}]_0 = \rho_0 \left[\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right]_0 \cdot \mathbf{a}. \tag{52}$$

We assume that internal stresses are equal to zero in the initial configuration. Thus we have to require $[\boldsymbol{\tau}]_0 = \mathbf{0}$, $[\boldsymbol{\mu}]_0 = \mathbf{0}$ and we conclude that

$$\left[\frac{\partial \mathcal{U}}{\partial \mathcal{E}}\right]_0 + \left[\frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}}\right]_0 \otimes \mathbf{m}_0 = \mathbf{0}, \quad \left[\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right]_0 \cdot \mathbf{a} = \mathbf{0}. \tag{53}$$

Taking (53), (50) into account we continue to expand (34):

$$[\boldsymbol{\tau}]_1 = \rho_0 \left(\left[\frac{\partial \mathcal{U}}{\partial \mathcal{E}}\right]_1 + \left[\frac{\partial \mathcal{U}}{\partial \mathcal{E}}\right]_0 \cdot \mathbf{g} + \left[\frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}}\right]_1 \otimes \mathbf{m}_0 + \left[\left(\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right)^{\top}\right]_0 \cdot \mathbf{f} \cdot \mathbf{a} \right), \tag{54}$$

$$[\boldsymbol{\mu}]_1 = \rho_0 \left(\left[\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right]_1 + \left[\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right]_0 \cdot \mathbf{g} \right) \cdot \mathbf{a}. \tag{55}$$

Assuming $\mathcal{U}$ to be sufficiently smooth in a neighbourhood of initial configuration, having done some calculations we obtain linear constitutive equations:

$$\begin{aligned} \boldsymbol{\tau} &= ({}^4\mathbf{X} \cdot\cdot\, \mathbf{g} + {}^4\mathbf{Y} \cdot\cdot\, \mathbf{f})^{\top}, \\ \boldsymbol{\mu} &= (\mathbf{g} \cdot\cdot\, {}^4\mathbf{Y} + {}^4\mathbf{Z} \cdot\cdot\, \mathbf{f})^{\top}, \end{aligned} \tag{56}$$

Here ${}^4\mathbf{X} = X_{mnkl}\mathbf{r}_m\mathbf{r}_n\mathbf{r}_k\mathbf{r}_l, {}^4\mathbf{Y} = Y_{mn\alpha l}\mathbf{r}_m\mathbf{r}_n\mathbf{r}_\alpha\mathbf{r}_l$, and ${}^4\mathbf{Z} = Z_{\alpha m\beta l}\mathbf{r}_\alpha\mathbf{r}_m\mathbf{r}_\beta\mathbf{r}_l$ are tensors of elastic constants:

$$
\begin{aligned}
{}^4\mathbf{X} = &\left[\frac{\partial^2 \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}^2}\right]_0 + \left[\frac{\partial}{\partial \boldsymbol{\mathcal{E}}}\otimes(\mathbf{m}_0 \frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}})\right]_0 + \mathbf{m}_0 \left[\frac{\partial}{\partial \boldsymbol{\gamma}}\frac{\partial \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}}\right]_0 + \mathbf{m}_0 \left[\frac{\partial}{\partial \boldsymbol{\gamma}}(\mathbf{m}_0 \otimes \frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}})\right]_0 + \\
&+ e_0 \mathbf{r}_k \otimes \mathbf{m}_0 \otimes \mathbf{r}_k \otimes \mathbf{m}_0, \\
{}^4\mathbf{Y} = &\left[\frac{\partial^2 \mathcal{U}}{\partial \varepsilon_{mn} \partial F_{k\alpha}}\right]_0 \mathbf{r}_m \otimes \mathbf{r}_n \otimes \mathbf{r}_\alpha \otimes \mathbf{r}_k + \mathbf{m}_0 \otimes \left[\frac{\partial^2 \mathcal{U}}{\partial \gamma_s \partial F_{k\alpha}}\right]_0 \mathbf{r}_s \otimes \mathbf{r}_\alpha \otimes \mathbf{r}_k + \\
&+ \mathbf{r}_\alpha \otimes \mathbf{m}_0 \otimes \mathbf{r}_\alpha \otimes \mathbf{f}_0, \\
{}^4\mathbf{Z} = &\left[\frac{\partial^2 \mathcal{U}}{\partial F_{m\alpha} \partial F_{k\beta}}\right]_0 \mathbf{r}_\alpha \otimes \mathbf{r}_m \otimes \mathbf{r}_\beta \otimes \mathbf{r}_k,
\end{aligned}
\tag{57}
$$

where $\left[\frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}}\right]_0 = -e_0 \mathbf{m}_0$, $\left[\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right]_0 = \mathbf{f}_0 \otimes \mathbf{m}_0$ (this can be obtained from (53) taking into account $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^\top$).

The linear approximation for 6th balance equation (47) coincides with linearized (48), because we assume that in the initial configuration $\overset{\circ}{\nabla}\mathbf{m}_0 = \mathbf{0}$, and hence the right-hand side of nonlinear 6th balance equation (43) is equal to zero in the linear approximation. Thus we see that linearized restrictions (47), (42) look as

$$
[\boldsymbol{\tau}_\times]_1 \cdot \mathbf{m}_0 = 0, \quad [\boldsymbol{\mu}]_1 \cdot \mathbf{m}_0 = 0. \tag{58}
$$

The restrictions on the tensors of elastic moduli following from (58) are:

$$
\begin{aligned}
\mathbf{m}_0 \cdot (\boldsymbol{\epsilon} \cdot\cdot {}^4\mathbf{X}) = \mathbf{0}, \\
\mathbf{m}_0 \cdot (\boldsymbol{\epsilon} \cdot\cdot {}^4\mathbf{Y}) = \mathbf{0},
\end{aligned}
\tag{59}
$$

where $\boldsymbol{\epsilon} = -\mathbf{E} \times \mathbf{E}$. One can see that tensors ${}^4\mathbf{X}, {}^4\mathbf{Y}, {}^4\mathbf{Z}$ satisfy (59), and there are no other restrictions for their components.

Thus we can state that stress and couple tensors in the linear theory are determined by

$$
[\boldsymbol{\tau}]_1 = \rho_0 \frac{\partial \mathcal{U}}{\partial \mathbf{g}}, \quad [\boldsymbol{\mu}]_1 = \rho_0 \frac{\partial \mathcal{U}}{\partial \mathbf{f}}, \tag{60}
$$

where the expression for strain energy in the linear theory is

$$
\begin{aligned}
\rho_0 \mathcal{U} = &\frac{1}{2}\mathbf{g} \cdot\cdot {}^4\mathbf{X} \cdot\cdot \mathbf{g} + \mathbf{g} \cdot\cdot {}^4\mathbf{Y} \cdot\cdot \mathbf{f} + \frac{1}{2}\mathbf{f} \cdot\cdot {}^4\mathbf{Z} \cdot\cdot \mathbf{f} = \\
= &\frac{1}{2}\left(\mathbf{g}^S \cdot\cdot {}^4\mathbf{T} \cdot\cdot \mathbf{g}^S + \mathbf{g}^A \cdot\cdot {}^4\mathbf{U} \cdot\cdot \mathbf{g}^A\right) + \mathbf{g}^S \cdot\cdot {}^4\mathbf{W} \cdot\cdot \mathbf{g}^A + \\
&+ \mathbf{g}^S \cdot\cdot {}^4\mathbf{H} \cdot\cdot \mathbf{f} + \mathbf{g}^A \cdot\cdot {}^4\mathbf{N} \cdot\cdot \mathbf{f} + \frac{1}{2}\mathbf{f} \cdot\cdot {}^4\mathbf{Z} \cdot\cdot \mathbf{f}.
\end{aligned}
\tag{61}
$$

Here $\mathbf{T}, \mathbf{U}, \mathbf{W}, \mathbf{H}, \mathbf{N}$ are tensors of elastic constants which are convenient to use. Tensors $\mathbf{Y} = \mathbf{H}+\mathbf{N}$, $\mathbf{U}$, and $\mathbf{W}$ are responsible for the coupling between angular and translational displacements.

We can find expressions (58)–(61) in [10], where the linear theory of medium under consideration is proposed.

From the formal point of view one can linearize (36) as well as (34) or any other variants of nonlinear constitutive equations obtained by the method described above for different systems of strain tensors. However it is much more simple to do for (34) or (39), because the linear terms of strain tensors (33) and (37) are not equal to zero and it is possible to go from (54), (55) to (56), (57).

2.5 Linear dynamic equations

The law of balance of momentum looks the same way as (17). To write down the law of balance of kinetic moment we must linearize the right-hand side of equation (19). Here $\boldsymbol{\tau}, \boldsymbol{\mu}$ are stress and couple tensors determined by (56). Using formulae (56), we get linear dynamic equations in displacements:

$$\begin{aligned} &\overset{\circ}{\nabla} \cdot ({}^4\mathbf{X} \cdot\cdot \mathbf{g} + {}^4\mathbf{Y} \cdot\cdot \mathbf{f})^\top + \rho\, \mathbf{Q} = \rho\, \ddot{\mathbf{u}}, \\ &\overset{\circ}{\nabla} \cdot (\mathbf{g} \cdot\cdot {}^4\mathbf{Y} + {}^4\mathbf{Z} \cdot\cdot \mathbf{f})^\top + ({}^4\mathbf{X} \cdot\cdot \mathbf{g} + {}^4\mathbf{Y} \cdot\cdot \mathbf{f})_\times^\top + \rho\, \mathbf{L} = \rho\, (\mu\, \ddot{\boldsymbol{\theta}} + \lambda\, \dot{\varphi}\, \dot{\boldsymbol{\theta}} \times \mathbf{m}_0). \end{aligned} \tag{62}$$

3 The analogy between Kelvin's medium, shells and ferromagnets

3.1 Elastic shells and Kelvin's medium

One may consider a non-classical elastic shell as a material surface every point of which is a "fibre" that can turn and move. Thus a shell is a 2D polar medium. Constitutive equations for shells can be obtained via the method described in [6]. As the own rotation of a fibre can not induce any stresses in the shell, the way to obtain constitutive equations of Kelvin's medium is exactly the same. Hence we can find an analogy between constitutive equations. In particular, for elastic shells one may use strain tensors (33) and constitutive equations (34), where ∇ and $\overset{\circ}{\nabla}$ are 2D nabla operators. It is more expedient to use instead of $\boldsymbol{\mathcal{E}}$ its 2D analog $(\overset{\circ}{\nabla}\mathbf{R} \cdot (\overset{\circ}{\nabla}\mathbf{R})^\top - \mathbf{a})/2$ in case of shells. Set of strain tensors (33) for shells describes deformations in a simpler way than in Kelvin's medium: in 2D case all components of strain tensors (33) are independent, and all independent kinds of deformation can be expressed as functions in these components, and in 3D case components of strain tensors (33) include all independent kinds of deformation, but some of these components are functions of others.

There are two differences between a shell and Kelvin's medium: 1) a shell is a 2D surface and Kelvin's medium is a 3D continuum; 2) every point body of Kelvin's medium (unlike a fibre of a shell) has a finite or large angular velocity of own rotation and non-zero axial moment of inertia λ. The second difference is essential when considering wave processes but does not influence the constitutive equations.

3.2 Ferromagnets and Kelvin's medium

3.2.1 Nonlinear equations

One example of deformable solids is saturated elastic ferromagnetic insulators. Below we will state some facts about ferromagnets which can be found in [7]. We will consider insulators to avoid necessity to take into account electric fields. We will not consider any processes concerning heat transfer though this restriction is not essential.

Every point of such a ferromagnet is characterized by displacement $\mathbf{u}$ and by density of a vector of magnetic moment $\mathbf{S}$. The state of magnetic saturation is defined as the state of ferromagnet when $|\mathbf{S}|$ is constant in radius-vector $\mathbf{R}$ and time t.

Forces and moments both of elastic and quantum mechanical nature act upon every point of a ferromagnet. The density of a moment induced by an external magnetic induction $\mathbf{B}^e$ is equal to

$$\mathbf{L} = \mathbf{S} \times \mathbf{B}^e. \tag{63}$$

Exchange interaction (interaction between spins depending on their relative turn) results in a close-range moment interaction; density of the above moment acting upon an elementary surface with the normal $\mathbf{n}$ is equal to

$$\mathbf{M}^{exc}_{(\mathbf{n})} = \mathbf{S} \times \boldsymbol{\Gamma}_{(\mathbf{n})} \tag{64}$$

where $\boldsymbol{\Gamma}_{(\mathbf{n})}$ is so called "contact exchange force". This allows to consider a tensor of exchange interactions $\boldsymbol{\mathcal{B}}$ such that

$$\mathbf{S} \times (\rho\, \boldsymbol{\Gamma}_{(\mathbf{n})} - \mathbf{n} \cdot \boldsymbol{\mathcal{B}}) = \mathbf{0}. \tag{65}$$

The power of a contact exchange force is equal to $\rho\, \boldsymbol{\Gamma}_{(\mathbf{n})} \cdot \dot{\mathbf{S}}$. Apart from the moment of exchange interaction, there exist forces of elastic nature and a "spin-lattice" moment. The latter depends on the direction of a magnetic moment of a point body relatively to the lattice.

The own kinetic moment of a point body in ferromagnet is equal to $\rho\, \mathbf{S}/\gamma$, where γ is the gyromagnetic ratio (constant).

The law of balance of force for elastic ferromagnetic insulator is

$$\frac{d}{dt} \int\limits_{\triangle V} \rho\, \mathbf{v}\, dV = \int\limits_{\triangle V} \rho\, \mathbf{Q}\, dV + \int\limits_{\Sigma} \boldsymbol{\tau}_{(\mathbf{n})}\, d\Sigma. \tag{66}$$

The law of balance of moment for elastic ferromagnetic insulator is

$$\frac{d}{dt} \int\limits_{\triangle V} \rho\, (\mathbf{r} \times \mathbf{v} + \mathbf{S}/\gamma)\, dV = \int\limits_{\triangle V} \rho\, (\mathbf{L} + \mathbf{R} \times \mathbf{Q})\, dV + \int\limits_{\Sigma} (\rho\, \mathbf{S} \times \boldsymbol{\Gamma}_{(\mathbf{n})} + \mathbf{R} \times \boldsymbol{\tau}_{(\mathbf{n})})\, d\Sigma. \tag{67}$$

The law of energy balance looks in the following way:

$$\frac{d}{dt} \int\limits_{\triangle V} \rho\, (\mathbf{v} \cdot \mathbf{v}/2 + \mathcal{U} - \mathbf{S} \cdot \mathbf{B}^e)\, dV =$$

$$= \int\limits_{\triangle V} \rho\, (\mathbf{Q} \cdot \mathbf{v} - \mathbf{S} \cdot \dot{\mathbf{B}}^e)\, dV + \int\limits_{\Sigma} (\rho\, \boldsymbol{\Gamma}_{(\mathbf{n})} \cdot \dot{\mathbf{S}} + \boldsymbol{\tau}_{(\mathbf{n})} \cdot \mathbf{v})\, d\Sigma. \tag{68}$$

For saturated ferromagnet $|\mathbf{S}| = \text{const}$, and one can write

$$\dot{\mathbf{S}} = \boldsymbol{\omega} \times \mathbf{S}. \tag{69}$$

Basing on (66)–(68) G.A. Maugin [7] obtains constitutive equations of ferromagnets using phenomenological approach.

Let us consider a saturated ferromagnet in terms of mechanics of elastic polar medium and interpret the facts stated above. It is a medium with interactions both of moment and force nature. Vector of translational displacement $\mathbf{u}$ and vector $\mathbf{S}$ are kinematic characteristics of every point body. Since $|\mathbf{S}| = \text{const}$, we may write $\mathbf{S} = \mathbf{P}(\mathbf{R}, t) \cdot \mathbf{S}_0$, where $\mathbf{S}_0$ is $\mathbf{S}$ in the initial configuration, and we can interpret $\boldsymbol{\omega}$ in equation (69) as angular velocity corresponding to the turn-tensor $\mathbf{P}$. One may use $\mathbf{P}$ as kinematic characteristic of a point body instead of $\mathbf{S}$. We may represent

$$\mathbf{P}(t) = \mathbf{P}_3(\psi \mathbf{m}_0) \cdot \mathbf{P}_2(\vartheta \mathbf{l}_0) \cdot \mathbf{P}_1(\varphi \mathbf{m}_0), \tag{70}$$

where $\mathbf{m}_0 = \mathbf{S}_0/|\mathbf{S}_0|$. Vector $\mathbf{S}$ does not depend on φ and no internal stresses can be induced by $\mathbf{P}_1$.

The power of an exchange force is $\rho\, \boldsymbol{\Gamma}_{(\mathbf{n})} \cdot \dot{\mathbf{S}} \overset{(69)}{=} \rho\, \boldsymbol{\Gamma}_{(\mathbf{n})} \cdot (\boldsymbol{\omega} \times \mathbf{S}) \overset{(64)}{=} \rho\, \mathbf{M}^{exc}_{(\mathbf{n})} \cdot \boldsymbol{\omega}$, i.e. it is equal to the power of an exchange moment performing mechanical work upon rotation of a point body with angular velocity $\boldsymbol{\omega}$. Thus, taking into account (69), (63), we rewrite the law (68) as

$$\frac{d}{dt} \int\limits_{\triangle V} \rho\, (\mathbf{v} \cdot \mathbf{v}/2 + \mathcal{U})\, dV = \\ = \int\limits_{\triangle V} \rho\, (\mathbf{Q} \cdot \mathbf{v} + \mathbf{L} \cdot \boldsymbol{\omega})\, dV + \int\limits_{\Sigma} (\rho\, \mathbf{M}^{exc}_{(\mathbf{n})} \cdot \boldsymbol{\omega} + \boldsymbol{\tau}_{(\mathbf{n})} \cdot \mathbf{v})\, d\Sigma. \tag{71}$$

Taking into account (64) – (67) and (69), we may write down the local form of (71):

$$\rho\, \dot{\mathcal{U}} = \boldsymbol{\tau} \cdot\cdot \nabla \mathbf{v} - \boldsymbol{\tau}_\times \cdot \boldsymbol{\omega} - (\mathcal{B} \times \mathbf{S}) \cdot\cdot \nabla \boldsymbol{\omega} \tag{72}$$

Comparing (25) and (72), we see that they coincide if we put

$$\boldsymbol{\mu} = -\mathcal{B} \times \mathbf{S} \equiv -S\, \mathcal{B} \times \mathbf{m}. \tag{73}$$

We can represent $\boldsymbol{\mu}$ in Kelvin's media in this way due to restriction (42).

We may try to draw a parallel between an elastic ferromagnetic insulator and Kelvin's medium. Let us consider medium with point bodies having infinitesimal moments of inertia with densities λ and μ but large angular velocity of own rotation $\dot{\varphi}$; vector $\mathbf{m}$ is an axis of a point body. In this case, the own kinetic moment of a particle is approximately equal to $\rho\, \lambda \dot{\varphi} \mathbf{m}$.

If the external moment has no projection on the axis of a point body $\mathbf{m}$ then $\dot{\varphi}$ is constant, and we can denote $S/\gamma = \lambda \dot{\varphi}$, $\mathbf{S} = S\mathbf{m}$. Thus we have $\rho\, \mathbf{S}/\gamma$ to be the kinetic moment of a point body, $|\mathbf{S}| = \text{const}$, and direction of $\mathbf{S}$ coincides with the axis of a particle.

Since we assume that $\mathbf{L}\cdot\mathbf{m}=0$, there exists vector $\mathbf{B}^e$ such that $\mathbf{L}=\mathbf{S}\times\mathbf{B}^e$, and $\mathbf{B}^e$ can be interpreted as external magnetic induction.

We may introduce the couple tensor $\boldsymbol{\mu}$ in a usual way. It has been shown (see section 2) that $\boldsymbol{\mu}\cdot\mathbf{m}=\mathbf{0}$. This allows us to represent $\boldsymbol{\mu}$ as $\boldsymbol{\mu}=-S\,\boldsymbol{\mathcal{B}}\times\mathbf{m}$ and to interpret $\boldsymbol{\mathcal{B}}$ as a tensor of exchange interaction.

If we rewrite the laws of balance of momentum, kinetic moment, and energy for Kelvin's medium in the new notation, we get these laws for saturated elastic ferromagnetic insulators that one can find in [7]. This allows us to obtain nonlinear constitutive equations for elastic ferromagnetic insulators with the method described above and the result will be exactly the same. Note that (33) (where $\tilde{\mathbf{a}}=\mathbf{E}\times\mathbf{m}_0$) can be found as an intermediate result in [7].

If we use (35) and suppose that $\mathcal{U}$ does not depend on α, we obtain the system of constitutive equations being used in the theory of ferromagnets. The assumption that $\dfrac{\partial\mathcal{U}}{\partial\alpha}=0$ demands that $\boldsymbol{\tau}_\times\cdot\mathbf{m}=0$, thus there exists a vector $\mathbf{B}^L$ such that $\boldsymbol{\tau}_\times=M\mathbf{B}^L\times\mathbf{m}$. In terms of the theory of ferromagnets, it means that a spin-lattice interaction is provided only by "local magnetic induction" ($\mathbf{B}^L$) acting upon spins. If we do not make the above assumption, then we have $\boldsymbol{\tau}_\times\cdot\mathbf{m}=\boldsymbol{\mu}^\top\cdot\cdot\nabla\mathbf{m}$ (see (43)). We see that when changing $\mathcal{U}(\boldsymbol{\mathcal{E}},\mathbf{F},\boldsymbol{\gamma})$ to $\mathcal{U}(\boldsymbol{\mathcal{E}},\boldsymbol{\Phi},\boldsymbol{\gamma})$ as G.A. Maugin does in [7], we lose the dependence on one of kinds of deformation (α), making the transformation dubious.

In [11] we can find that representation for $\mathcal{U}$ may include the term $\mathbf{m}\cdot(\nabla\times\mathbf{m})=\mathrm{tr}(\mathbf{K}\cdot\mathbf{a}\cdot\mathbf{A}^{-1})$. To ensure that $\boldsymbol{\tau}_\times\cdot\mathbf{m}=0$ (as G.A. Maugin [7] requires) we must set $\mathcal{U}=\mathcal{U}(\boldsymbol{\mathcal{E}},\boldsymbol{\Phi},\boldsymbol{\gamma})$, i.e. exclude dependence $\mathcal{U}$ on $\mathbf{F}$. However, it is impossible to satisfy both of these requirements, because $\mathbf{F}=\mathbf{K}\cdot\mathbf{a}\cdot\mathbf{A}^{-1}\cdot(2\boldsymbol{\mathcal{E}}-\boldsymbol{\gamma}\otimes\boldsymbol{\gamma}+\mathbf{E})$.

Thus we have an exact analogy between ferromagnets and Kelvin's medium. An axis of a point body corresponds to the unit vector of a spin, all angular characteristics correspond to magnetic subsystem and translational ones to elastic subsystem. We make the following analogies:

$\mathbf{u}$ is the translational displacement in both media;

$\mathbf{m}$ is the axis of a point body in a Kelvin's medium and the unit vector of a magnetic moment (or of a spin) in a ferromagnet;

$\boldsymbol{\tau}$ is stress tensor in both media;

$\boldsymbol{\mu}=-\boldsymbol{\mathcal{B}}\times\mathbf{S}$ is the couple tensor; $\boldsymbol{\mathcal{B}}$ is the tensor of exchange interactions;

$\rho\,\lambda\dot{\varphi}\mathbf{m}=\rho\,\mathbf{S}/\gamma$ is the kinetic moment, where λ is the density of axial moment of inertia, $\dot{\varphi}$ is the angular velocity of own rotation in Kelvin's medium; $\mathbf{S}$ is the magnetic moment, γ is the gyromagnetic ratio, $\rho\,S=M$ is the magnetization in ferromagnet.

[Note that λ needs to be infinitesimal and $\dot{\varphi}$ large for the analogy to work];

$\mathbf{L}=\mathbf{B}^e\times\mathbf{m}$ is the density of external body moment; $\mathbf{B}^e$ is the external magnetic induction;

$\boldsymbol{\tau}_\times=M\mathbf{B}^L\times\mathbf{m}$; $\mathbf{B}^L$ is the local magnetic induction in ferromagnet [only if $\mathcal{U}$ does not depend on α].

Exchange interaction corresponds in Kelvin's medium to a moment acting upon a particle depending on the relative turn of particles; $\boldsymbol{\tau}_\times$ (and $\mathbf{B}^L$ respectively), namely spin-lattice interaction, correspond to a moment depending only on the orientation of the particle under consideration (as if all other point bodies were point masses).

The coupling between angular and translational displacements in Kelvin's medium corresponds to the magnetoacoustic phenomena. Therefore it seems to be very important to properly take into account the dependence of $\mathcal{U}$ on the "mixed" kinds of deformations such as α (if we use (35)) or $\mathbf{F}$ (if we use (33)). These phenomena are most interesting both from theoretical and practical points of view.

3.2.2 Linear equations

To obtain linear equations it is more convenient to use (33) as opposed to (35) because tensor $\mathbf{\Phi} = o(\theta^2)$ when $\theta = o(1)$. G.A. Maugin [7] does not follow this way and his results are different from (56). If we put in (56) ${}^4\mathbf{Y} = 0$, we get equations linearized relatively to ferromagnetic phase obtained in [7]. We can see that our expression is more general because it includes the term coupling elastic, spin-lattice and exchange interactions. This coupling occurs in real magnetic solids, and sometimes it results in formation of helicoidal magnetic structures [11]. We can suppose that taking this term into account is significant for description of magnetoacoustic resonance.

In case of ferromagnets in (61) tensor ${}^4\mathbf{T}$ is the tensor of elastic constants, ${}^4\mathbf{U}$ depends on the magnetizability constants, ${}^4\mathbf{W}$ can be expressed through piezomagnetic constants, and ${}^4\mathbf{Z}$ can be expressed through ferromagnetic exchange constants.

Linear dynamic equations are the same as (62) but we have to put $\mu = 0$ in the right-hand side of the second equation since the analogy between dynamic equations of ferromagnets and Kelvin's medium exists if λ, μ are infinitesimal and $\dot{\varphi}$ is large. Setting $\mu = 0$, we lose the influence of the initial conditions.

Let us consider the case when the external magnetic induction $\mathbf{B}^e$ can be expressed as $B_0\mathbf{m}_0 + \tilde{\mathbf{B}}$, where $\tilde{\mathbf{B}}$ is infinitesimal and $\tilde{\mathbf{B}} \cdot \mathbf{m}_0 = 0$. Then external moment $\mathbf{L}$ can be written as

$$\mathbf{L} = S\mathbf{m} \times \mathbf{B}^e = S(\mathbf{m}_0 + \theta \times \mathbf{m}_0) \times (B_0\mathbf{m}_0 + \tilde{\mathbf{B}}) + O\,(\theta \cdot \tilde{\mathbf{B}}) = S(\mathbf{m}_0 \times \tilde{\mathbf{B}} - B_0\theta) + O\,(\theta \cdot \tilde{\mathbf{B}})$$

We obtain linear dynamic equations:

$$\begin{aligned}
&\overset{\circ}{\nabla} \cdot ({}^4\mathbf{X} \cdot\cdot (\overset{\circ}{\nabla}\mathbf{u} + \theta \times \mathbf{E}) + {}^4\mathbf{Y} \cdot\cdot \overset{\circ}{\nabla}\theta)^\top + \rho\,\mathbf{Q} = \rho\,\ddot{\mathbf{u}},\\
&\overset{\circ}{\nabla} \cdot ((\overset{\circ}{\nabla}\mathbf{u} + \theta \times \mathbf{E}) \cdot\cdot {}^4\mathbf{Y} + {}^4\mathbf{Z} \cdot\cdot \overset{\circ}{\nabla}\theta)^\top + \\
&\quad + ({}^4\mathbf{X} \cdot\cdot (\overset{\circ}{\nabla}\mathbf{u} + \theta \times \mathbf{E}) + {}^4\mathbf{Y} \cdot\cdot \overset{\circ}{\nabla}\theta)^\top_\times + M\mathbf{m}_0 \times \tilde{\mathbf{B}} - MB_0\theta = M\dot{\theta} \times \mathbf{m}_0.
\end{aligned} \tag{74}$$

One can see that B_0 acts as a torsional spring.

4 Wave processes

We shall consider wave propagation in the medium with infinitesimal angles of nutation for the case when exact analogy between ferromagnets and Kelvin's medium can be established, i.e. λ, μ are infinitesimal, $\dot{\varphi}$ is large, $\lambda\dot{\varphi} = O(1)$. The results can be interpreted both in terms of Kelvin's medium and ferromagnets.

Let us consider the case $\mathbf{Q} = 0$, $\tilde{\mathbf{B}} = 0$. We shall search for a solution of (74) in the form $\mathbf{u} = \mathbf{u}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\Omega t)}$, $\boldsymbol{\theta} = \boldsymbol{\theta}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\Omega t)}$. After substituting into (62) we obtain a spectral problem:

$$\begin{aligned}
&({}^4\mathbf{X}_1 \cdot\cdot \mathbf{k}\otimes\mathbf{k} - \rho\,\Omega^2\,\mathbf{E}) \cdot \mathbf{u}_0 + (i\,\mathbf{k} \cdot {}^4\mathbf{X}_3 \cdot\cdot \boldsymbol{\epsilon} + {}^4\mathbf{Y}_1 \cdot\cdot \mathbf{k}\otimes\mathbf{k}) \cdot \boldsymbol{\theta}_0 = 0,\\
&({}^4\mathbf{Y}_2 \cdot\cdot \mathbf{k}\otimes\mathbf{k} - i\,\boldsymbol{\epsilon} \cdot\cdot {}^4\mathbf{X} \cdot \mathbf{k}) \cdot \mathbf{u}_0 + \\
&+ ({}^4\mathbf{Z}_1 \cdot\cdot \mathbf{k}\otimes\mathbf{k} + \boldsymbol{\epsilon} \cdot\cdot {}^4\mathbf{X} \cdot\cdot \boldsymbol{\epsilon} + MB_0\mathbf{a} + i\,(2({}^3\tilde{\mathbf{N}} \cdot \mathbf{k})^A + M\mathbf{m}_0 \times \mathbf{E})) \cdot \boldsymbol{\theta}_0 = 0,
\end{aligned} \tag{75}$$

where

$$\begin{aligned}
{}^4\mathbf{X}_1 &= X^{mnkl}\,\mathbf{r}_m\otimes\mathbf{r}_k\otimes\mathbf{r}_l\otimes\mathbf{r}_n, & {}^4\mathbf{X}_3 &= X^{mnkl}\mathbf{r}_n\otimes\mathbf{r}_m\otimes\mathbf{r}_k\otimes\mathbf{r}_l, & (76)\\
{}^4\mathbf{Y}_1 &= Y^{mn\beta l}\mathbf{r}_m\otimes\mathbf{r}_\beta\otimes\mathbf{r}_l\otimes\mathbf{r}_n, & {}^4\mathbf{Y}_2 &= Y^{mnkl}\,\mathbf{r}_k\otimes\mathbf{r}_m\otimes\mathbf{r}_n\otimes\mathbf{r}_l, & (77)\\
{}^3\tilde{\mathbf{N}} &= -\boldsymbol{\epsilon} \cdot\cdot {}^4\mathbf{N}, & {}^4\mathbf{Z}_1 &= Z^{\alpha n\beta l}\,\mathbf{r}_\alpha\otimes\mathbf{r}_\beta\otimes\mathbf{r}_l\otimes\mathbf{r}_n. & (78)
\end{aligned}$$

Let us consider a material with a transversal isotropy (for highly symmetric media we have $\mathbf{Y} = 0$). Let $\mathbf{m}_0$ be an axis of isotropy. We shall investigate the particular case when $\mathbf{X}$ is isotropic, and $\mathbf{Z}$ is orthotropic. In this case taking into account restrictions (59) given by 6th balance equation in the linear theory (48) we have

$${}^4\mathbf{X} = (X^{11} - X^{22})\mathbf{E}\otimes\mathbf{E} + 2X^{22}(\mathbf{r}_m\otimes\mathbf{r}_n)^S(\mathbf{r}^m\otimes\mathbf{r}^n)^S \tag{79}$$

$$\mathbf{H} = H^{11}\mathbf{a}\otimes\mathbf{a} + H^{22}(\mathbf{a}_2\otimes\mathbf{a}_2 + \mathbf{a}_4\otimes\mathbf{a}_4), \tag{80}$$

where $\mathbf{a}_2 = \mathbf{r}_1\otimes\mathbf{r}_1 - \mathbf{r}_2\otimes\mathbf{r}_2$ and $\mathbf{a}_4 = \mathbf{r}_1\otimes\mathbf{r}_2 + \mathbf{r}_2\otimes\mathbf{r}_1$

$$\mathbf{N} = N((\mathbf{r}_2\otimes\mathbf{r}_3)^A\otimes\mathbf{r}_1\otimes\mathbf{r}_3 + (\mathbf{r}_3\otimes\mathbf{r}_1)^A\otimes\mathbf{r}_2\otimes\mathbf{r}_3), \tag{81}$$

$$\begin{aligned}
\mathbf{Z} =& Z^{11}\mathbf{a}\otimes\mathbf{a} + Z^{33}\mathbf{a}_3\otimes\mathbf{a}_3 + Z^{22}(\mathbf{a}_2\otimes\mathbf{a}_2 + \mathbf{a}_4\otimes\mathbf{a}_4) + \\
&+ Z^{1313}(\mathbf{r}_1\otimes\mathbf{r}_3\otimes\mathbf{r}_1\otimes\mathbf{r}_3 + \mathbf{r}_2\otimes\mathbf{r}_3\otimes\mathbf{r}_2\otimes\mathbf{r}_3)
\end{aligned} \tag{82}$$

where $\mathbf{a}_3 = \mathbf{r}_1\mathbf{r}_2 - \mathbf{r}_2\mathbf{r}_1$, and (75) can be rewritten as

$$\begin{pmatrix}
A_{11} - \rho\,\Omega^2 & A_{12} & A_{13} & B_{11} & B_{12}\\
A_{12} & A_{22} - \rho\,\Omega^2 & A_{23} & B_{21} & B_{22}\\
A_{13} & A_{23} & A_{33} - \rho\,\Omega^2 & B_{31} & B_{32}\\
B_{11} & B_{21} & B_{31} & C_{11} & C_{12} - iM\Omega\\
B_{12} & B_{22} & B_{32} & C_{21} + iM\Omega & C_{22}
\end{pmatrix} \cdot \begin{pmatrix} u_1\\ u_2\\ u_3\\ \theta_1\\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0 \end{pmatrix} \tag{83}$$

where $A_{mn}, B_{m\beta}, C_{\alpha\beta}$ are real polynomials of second degree in $\mathbf{k}$ (see Appendix A). A_{mn} depend only on $\mathbf{X}$ and $B_{m\beta}$ depend on $\mathbf{Y}$, $C_{\alpha\beta}$ depend on $\mathbf{Z}, M$, and B_0.

If $\mathbf{Y} = \mathbf{0}$, i.e. there is no coupling between angular and translational displacements (magnetic and elastic subsystems), possible dispersion relation graphs are shown in Fig. 2. There exists a cut-off frequency for angular (spin) waves. The position of the curve

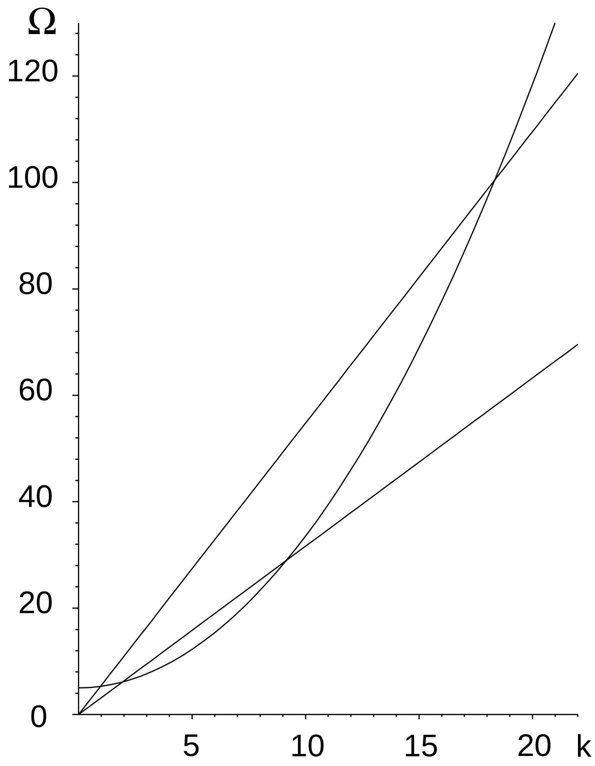

Figure 2: Partial dispersion curves

corresponding to these waves depends on magnetization M (or on the kinetic moment of a point body in Kelvin's medium); the curve may lay higher than other curves if M is not large enough. The cut-off frequency increases with increasing B_0. [Curves in Fig. 2 are calculated with parameters $M = 50$, $B_0 = 5$, $X^{11} = 20$, $X^{22} = 10$, $Z^{11} = 20$, $Z^{22} = 10$, $\mathbf{k} \cdot \mathbf{m}_0 = 0$].

Let us consider the case when $\mathbf{Y}$ is infinitesimal, i.e. the coupling between angular and translational displacements in Kelvin's medium (magnetic and elastic subsystems in ferromagnet) is weak.

The graphs for dispersion relations are close to the partial curves corresponding to the case of independent oscillations of elastic and magnetic subsystems. However, there is an essential difference: the coupling, even weak, qualitatively changes the behaviour of the curves in neighbourhoods of their intersections.

Indeed, suppose $B_{m\beta} \neq 0$ but infinitesimal. Let us consider equation (83) in possibly complex coordinates such that matrixes A and C are diagonal. After these changes of variables components of matrix B continue to be infinitesimal. Suppose the point (Ω^*, k^*) is an intersection of two graphs for partial dispersion equations. Consider a particular case when these roots of partial dispersion equations are not multiple. It means $f_1(\Omega^*, k^*) = f_2(\Omega^*, k^*) = 0$, where $f_1(\Omega, k)$ and $f_2(\Omega, k)$ are diagonal components of matrix after transformation, $k = |\mathbf{k}|$. In a neighbourhood of (Ω^*, k^*) problem (83) may

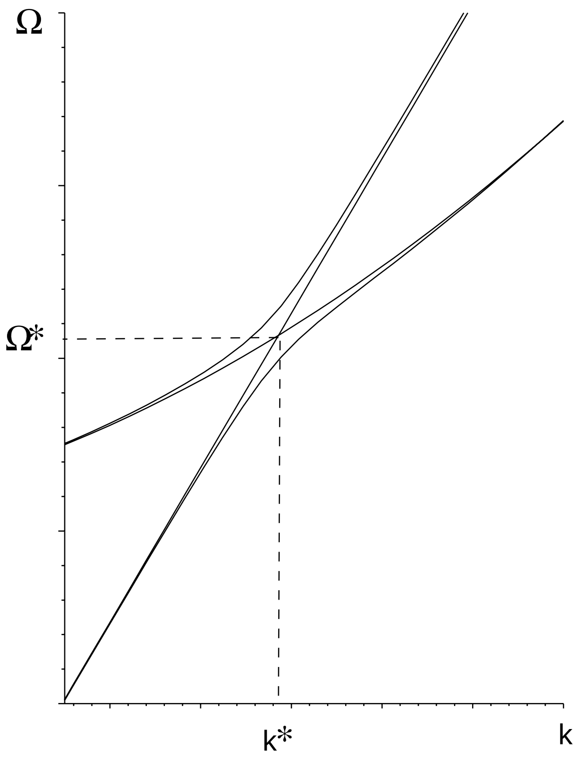

Figure 3: Dispersion curves for partial and coupled waves

be written as follows:

$$f_1(\Omega, k)f_2(\Omega, k) - b(\Omega, k) = 0, \tag{84}$$

where b is infinitesimal and depends on the other matrix components. Supposing that $b(\Omega^*, k^*) = b_* \neq 0$ (the coupling is not degenerative at this point) we shall expand f_1 and f_2. Let $\tilde{\Omega} = \Omega - \Omega^*$, $\tilde{k} = k - k^*$, $c_{g1} = -\frac{f'_{1k}}{f'_{1\Omega}}$, $c_{g2} = -\frac{f'_{2k}}{f'_{2\Omega}}$ (group velocities of non-coupled waves). Expanding to second order terms, (84) becomes

$$(\tilde{\Omega} - \tilde{k}(c_{g1} + c_{g2})/2)^2 - \tilde{k}^2(c_{g1} - c_{g2})^2/4 = b_*. \tag{85}$$

It is a hyperbola with asymptotes coinciding with tangents to the partial curves $f_1 = 0$, $f_2 = 0$ at the point (Ω^*, k^*). We see that curves corresponding to the weak coupled waves have no intersection in the neighbourhood of (Ω^*, k^*); they are close to each other and their group velocities are equal at k^* and at Ω^* (Fig. 3).

This behaviour of the dispersion relation graphs points to the phenomenon called magnetoacoustic resonance in the physics of ferromagnets. Elastic and spin waves interact with each other in the neighbourhood of this point; for example, it is possible to excite elastic waves with an external magnetic field or to induce a spin wave with an elastic one. This phenomenon has a lot of applications in technology.

Since we take into account couplings of elastic and spin waves in the most general way, we see more principal possibilities for existence of phenomena akin to magnetoacoustic resonance. In the material under consideration there can be four points analogous to the point of magnetoacoustic resonance (the number of this points depends on the direction of $\mathbf{k}$). The intersection of compression wave graph and magnetic waves corresponds to the case described above. The intersection of shear wave graphs and magnetic wave graphs needs a separate analysis since partial shear curve corresponds to a double root in the case of isotropic $\mathbf{X}$.

Graphs in Fig. 4 are calculated with parameters $\mathsf{M} = 50, \mathsf{B}_0 = 5, \mathsf{X}^{11} = 20, \mathsf{X}^{22} = 10$, $\mathsf{Z}^{11} = 20, \mathsf{Z}^{22} = 10, \mathsf{H}^{11} = \mathsf{H}^{22} = 1$, $\mathbf{k} \cdot \mathbf{m}_0 = 0$, and in Fig. 2 with the same parameters but ${}^4\mathbf{H} = \mathbf{0}$. In the latter case all resonances disappear. One can see that the assumption of G.A. Maugin ${}^4\mathbf{Y} = \mathbf{0}$ may exclude from consideration some essential phenomena.

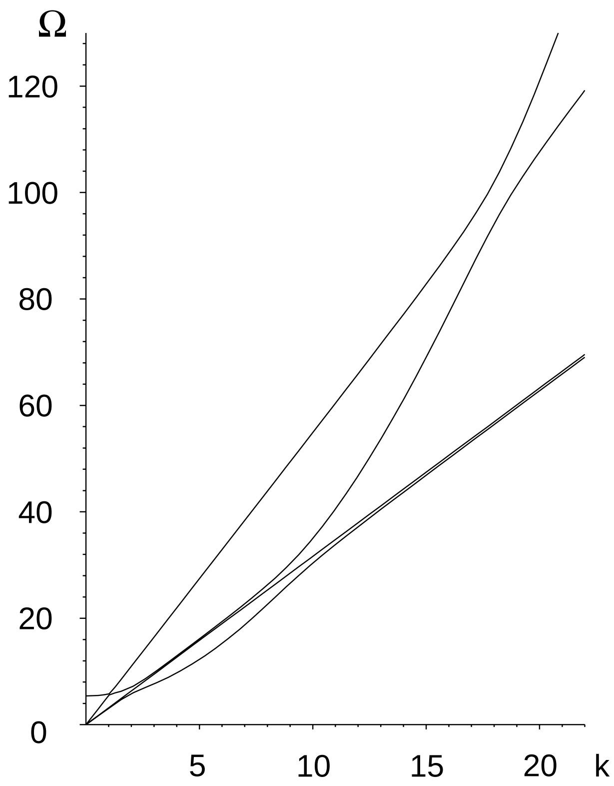

Figure 4: Dispersion curves: four resonances

To understand clearly these phenomena one has to consider nonlinear theory. The most difficult problem is to find the concrete form of nonlinear strain energy. We have seen that it is very important to take into account all kinds of deformation that can provide coupling between elastic and magnetic subsystem.

5 Conclusion

In this paper we obtain nonlinear constitutive and dynamic equations of Kelvin's medium. We show an exact analogy between elastic ferromagnetic insulators and this medium. We consider all kinds of deformations that can induce internal stresses, which gives the possibility to take into account the interaction between magnetic and elastic subsystems in the most general way. This is important for description of a magnetoacoustic resonance. For example, there can be found more resonances for some directions of wave propagation in low symmetric materials.

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Appendix A

$$
\begin{aligned}
A_{11} &= k_1^2(X^{11} + X^{22}) + (k_2^2 + k_3^2)X^{22} \\
A_{22} &= k_2^2(X^{11} + X^{22}) + (k_1^2 + k_3^2)X^{22} \\
A_{33} &= k_3^2(X^{11} + X^{22}) + (k_1^2 + k_2^2)X^{22} \\
A_{12} &= k_1 k_2 X^{11} \\
A_{13} &= k_1 k_3 X^{11} \\
A_{23} &= k_2 k_3 X^{11} \\
B_{11} &= k_1^2(H^{11} + H^{22}) + k_2^2 H^{22} \\
B_{22} &= k_2^2(H^{11} + H^{22}) + k_1^2 H^{22} \\
B_{12} = B_{21} &= k_1^2 k_2^2 H^{11} + k_3^2 N/2 \\
B_{31} &= -k_2 k_3 N/2 \\
B_{32} &= -k_1 k_3 N/2 \\
C_{11} &= k_1^2(Z^{11} + Z^{22}) + k_2^2(Z^{22} + Z^{33}) + k_3^2 Z^{1313} + MB_0 \\
C_{22} &= k_2^2(Z^{11} + Z^{22}) + k_1^2(Z^{22} + Z^{33}) + k_3^2 Z^{1313} + MB_0 \\
C_{12} &= k_1 k_2(Z^{11} - Z^{33})
\end{aligned}
\qquad (86)
$$

On the Painleve Paradoxes*

1 Introduction

The friction is one of the most widespread phenomena in a Nature. The manifestations of friction are rather diverse. The laws, with which the friction in concrete situations is described, are diverse as well. Most popular in the applications are two laws of friction: the linear law of viscous friction and so-called dry friction. The viscous friction is well investigated, and its manifestations are clear and are easily predicted. It cannot be said about the laws of dry friction, though they are studied and are used in practice already more than two hundred years. Note that the friction, arising at sliding of one rigid body on another at absence of greasing, is called the dry friction. The relative sliding of bodies in contact, as a rule, is accompanied by occurrence of forces of friction, which render significant influence on dynamic processes in different sorts technical devices. Coulomb carried out the first researches of the dry friction in the end of XVIII century. The schematic of the Coulomb experiment is submitted in a Fig.1.

In 1791 Coulomb has published the first formulation of the law of dry friction in the following simple form.

$$F_{fr} = -\mu N \operatorname{sign} \dot{x}, \quad \text{if} \quad \dot{x} \neq 0, \tag{A}$$

The external simplicity of this law rather deceptive. As a matter of fact the Coulomb law of friction is the most complicated constitutive equation in mechanics. This may be seen, for example, from the fact that up to now the general mathematical statement of the Coulomb law of friction is absent in literature. The formulation (A) is only small part of general statement. In experiments by Coulomb the force of squeezing N of bodies was set and was known. However, this force is not known in the most of nontrivial problems and must be found in the process of a solution of the considered task. In some cases, the function $N(t)$ can have complex view and depends on many physical features of the task under consideration. Factor of friction μ is accepted to be the characteristic of bodies in contact. Now factors of friction for various pairs of bodies are resulted in the data books. In the simple situations the Coulomb law allows completely to solve the put task. During about one century it was considered, that the Coulomb law does not comprise any ambiguities from the theoretical point of view. At the same time,

*Wiercigroch M., Zhilin P.A. On the Painleve Paradoxes // Proceedings of the XXVII Summer School "Nonlinear Oscillations in Mechanical Systems", St. Petersburg, Russia, 2000, P. 1–22.

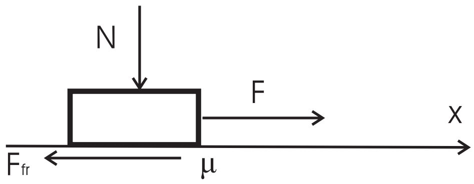

Figure 1: The Coulomb experiment

the rough development of machine-tool construction in second half of XIX century has revealed many cases, in which, on the first sight, the application of the Coulomb law leads to some contradictions. The special anxiety was caused by strange vibrations of machine tools (in some decades they were investigated and have received the name of frictional self-oscillations), processing, sharply lowering accuracy, of let out products. Sometimes the character of the movements arising in certain conditions was very strange, almost saltatory. Now such saltatory movements became object of intensive researches — see [1]. These circumstances, and also the theoretical needs, have forced the researchers again to address to the Coulomb law of dry friction. In 1895 Painleve has published the controversial book [2]. In what follows we shall cite the book [3], which contain other important works [4]–[14] on the subject. In [12] the opinion was expressed, that the Coulomb law is incompatible to the basic principles of the mechanics. Analyzing numerous examples of application of the Coulomb law in tasks of dynamics of systems with friction, Painleve comes to completely unexpected conclusion: "... While the marked special conditions are carried out, law by Coulomb is in the contradiction with dynamics of rigid bodies" [12], (see [3], p. 246) and further "... Between dynamics of a rigid body and the Coulomb law there is a logic contradiction under conditions, which can be carried out in the reality" [3], p. 248. As the logic a contradiction Painleve names situations, when the solution of the basic task of dynamics either does not exist, or is not unique. In modern literature these contradiction are known as the Painleve paradoxes. Now conclusions by Painleve even if they would be completely correct already anybody would not surprise. In continuum mechanics there is a chapter devoted to the theory of the constitutive equations, where the basic problem is the statement of conditions, at which those or other constitutive equations lead to the correctly put tasks. The Coulomb law is the typical constitutive equation, which, basically, can appear unacceptable. The merit of Painleve consists that he was the first who has pointed out at this central problem in mechanics. The Painleve results have called forth long discussion, in which such scientists as L. Prandtl, F. Klein, R. Von Mises, G. Hamel, L. Lecornu, de Sparre, F. Pfeifer and, of course, P. Painleve have taken part. The opinions of the participants of discussion were separated. L. Lecornu [7, 8], in essence, having recognized presence of paradoxes, offers to refuse from the model of rigid body. F. Klein [6] has come to a conclusion: "The Coulomb law of friction is not in the contradiction neither with principles of mechanics, nor with the phenomena observable in a nature: they need only correctly to be interpreted". An originality of results by F. Klein is caused by that he for the first time in tasks of a considered type used "hypothesis" of the instant stopping.

In this occasion the discussion has found new features, and at its center there was a hypothesis of F. Klein, which F. Klein did not consider as a hypothesis, but also has not deduced it on a level of a fact in evidence. R. Von Mises [9] concerning a hypothesis of F. Klein has expressed so: “1. F. Klein explains the phenomenon not from the point of view of the Coulomb law, but using a new rule obtained from experience. 2. This new skilled rule can be represented in the form of some modification of the Coulomb law”. Further R. Von Mises results rather interesting reasons and gives the formulation adding the Coulomb law and allowing to combine sights of Painleve and Klein. Nevertheless, final conclusion by R. Von Mises is those: “Thus, not logic, but the methodology of the Newton mechanics compels us to refuse from the Coulomb law”. G. Hamel [5] has joined the point of view by L. Lecornu about failure of the rigid body model. L. Prandtl [14] has expressed rather definitely: “In the statements of Mises and Hamel the speech goes about” to a hypothesis “of instant stopping. As opposed to this I emphasize, that in this case it is possible to speak only about result obtained through limiting transition. The research of elastic systems shows, generally speaking, something greater: it may be established, that from two possible movements, which the conventional theory gives for positive pulses, one, namely, accelerated motion will be steady, and another, slowed down, will be, on the contrary, unstable. In a limit we obtain the indefinitely large instability. So it is quietly possible to tell, that this second movement is practically impossible. From this it follows, that it is impossible by no means to expose of logic doubts against the Coulomb law”. Under the Prandtl offer, F. Pfeifer made the large research [13]. However, the clear confirmation of such point of view was not carried out. Thus, in discussion the Painleve position has not found a convincing refutation, as was marked in three notes by Painleve [10, 11, 12] during the discussion. Even those authors, which disagree with the Painleve position, have not specified in which items of the Painleve reasoning is mistaken, and, hence, the position of Painleve remains not challenged. There was an opinion, which P. Appell [15], p. 117, has expressed in the following words: “it is not necessary to think, that only in exclusive cases there can be possible such difficulties. On the contrary, they arise in the most common cases, at least, at enough large value of factor of friction μ. Because of this new experiments for a finding of the laws of friction, which is not resulting more in these difficulties, are necessary”. Nevertheless, some ways of an exit from paradoxical situations were shown. The basic way of an exit is refusal of the rigid body model. Other way is application if necessary “hypotheses” of the instant stopping. However, its substantiation remained behind frameworks of the carried out researches. For decades, past from time of end of discussion, the interest to the Painleve paradoxes that faded, again grew. N.V. Butenin [16] showed fruitfulness of the Klein hypothesis in the large work. The significant development of ideas connected to partial refusal of the rigid body model was made in works of Le Suan Anh [17], in which the references to many other works can be found.

From told follows that it is necessary, firstly, to show features of the Coulomb law of friction, not complicated by any other circumstances, and, secondly, it is necessary to consider those conditions, which were investigated by Painleve. Only after that it will be possible either to recognize a position by Painleve, or to reject it partially or completely. It is well known that the tasks with the Coulomb friction have the not unique solution even in the elementary cases. F. Klein marked the importance of this fact for the first time. Namely, F. Klein has found out the existence of discontinues solutions, which should be taken into account for avoidance of the Painleve paradoxes. However majority of the scientists have not accepted the result of F. Klein. It is easy to understand the main reason of this. In the offer by Klein we deal with instant stopping of a body of nonzero weight. It is well known, that in such a case it is necessary to apply the infinitely large force, what is impossible in a reality. In works [8, 17] the physical sense of the discontinues solutions was shown and is specified as to choose the necessary solution from two possible ones. Nevertheless, as it became clear from the subsequent discussions, there is a necessity to consider the solution by F. Klein more carefully.

In given paper the authors are going to show the following. The authors agree that the laws of dry friction, similarly to all experimentally established laws, require the further researches and specifications. It is necessary, for example, if we wish to construct the satisfactory theory of frictional auto vibrations. At the same time, the authors resolutely object to the established opinion that the law of friction by Coulomb is the reason of certain paradoxical results contradicting to the experimental facts or common sense. If to consider cases, known in the literatures under the name of the Painleve paradoxes, then it is easy to see that all of them concern to dynamic tasks for systems of rigid bodies. It is well known that these tasks very frequently appear incorrectly put, though the law of friction by Coulomb in them can not be applied. Nonuniqueness or nonexisting of the solution are typical manifestations of the incorrectly put tasks. If we want to work with rigid bodies, we should be ready that the not unique solutions can appear which, in addition, can be non-smooth. The question, hence, consists not in getting rid of them, but in giving them correct interpretation. The significant part of given paper is devoted to this. Let's note, that in tasks of dynamics of systems with the Coulomb friction frequently shows features, characteristic for dynamics of systems at shock loading. Sometimes this shock loading appears larvae. Let's show told on an example of a task shown in a Fig. 1. We assume, that the body moved at $t < 0$ with constant speed. At the moment of time $t = 0$ all active forces stop the action, and the body goes on inertia. Actually at t=0 occurs shock loading of a body by force of friction. Really, at $t < 0$ on a body any forces did not act, as the active force was counterbalanced by force of friction. When the active force has disappeared, the shock loading of a body by force of friction has taken place. In other words, the collision of rigid bodies has taken place at absence of seen attributes of impact.

2 The Coulomb Law of Friction

The conventional formulation of the Coulomb law of dry friction in textbooks has a form

$$F_{fr} = -\mu N \operatorname{sign} \dot{x}, \quad \text{if} \quad \dot{x} \neq 0, \tag{1}$$

where the notation of the Fig.1 are used. Let us consider the task shown on the Fig.2 Making use Eq.(1) one may write the next equation of motion

$$m\ddot{x} + \mu m g \operatorname{sign}\left(\dot{x} - x_0 \omega \cos \omega t\right) = 0. \tag{2}$$

Initial conditions have a form

$$t = 0: \quad x = 0, \quad \dot{x} = 0. \tag{3}$$

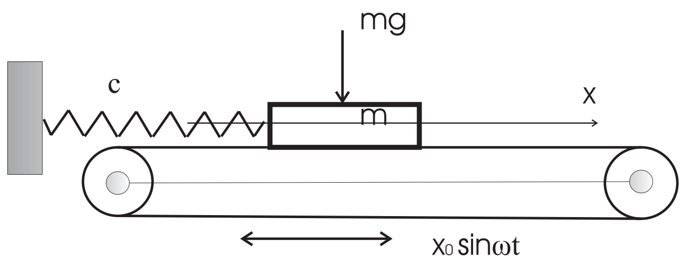

Figure 2: The body on vibro-transveyer

The Cauchy problem (2)–(3) may be solved, but its solution will not correspond to the real motion of the mass m. The reason is that the equality (1) expresses only part of the Coulomb low of dry friction. The general statement of this law can be represented in the form

$$F_{fr} = \begin{cases} -\mu N \operatorname{sign} \dot{x}, & \text{if} \quad \dot{x} \neq 0, \\ f_{st},\ |f_{st}| \leqslant \mu N & \text{if} \quad \dot{x} = 0, \end{cases} \tag{4}$$

where f_{st} must be determined from the static equation. More exact expression of the Coulomb law of friction is given by representation

$$F_{fr} = \begin{cases} -\mu N \operatorname{sign} \dot{x}, & \text{if} \quad \tau^2 \ddot{x}^2 + \dot{x}^2 \neq 0, \\ f_{st},\ |f_{st}| \leqslant \mu N & \text{if} \quad \tau^2 \ddot{x}^2 + \dot{x}^2 = 0, \end{cases} \tag{5}$$

where τ is the time-like parameter. One has to remember that the force of squeezing N must be nonnegative $N \geqslant 0$. If we have the two-sided contact then equality (5) must be replaced by the expression

$$F_{fr} = \begin{cases} -(\mu_1 N_1 + \mu_2 N_2) \operatorname{sign} \dot{x}, & \text{if} \quad \tau^2 \ddot{x}^2 + \dot{x}^2 \neq 0, \\ f_{st},\ |f_{st}| \leqslant \mu_1 N_1 + \mu_2 N_2, & \text{if} \quad \tau^2 \ddot{x}^2 + \dot{x}^2 = 0, \end{cases}, \quad \begin{array}{l} N_1 \geqslant 0, \\ N_2 \geqslant 0, \end{array} \tag{6}$$

where $\mu_1,\ \mu_2$ are the factors of friction of downside and upside of the contact respectively, $N_1,\ N_2$ are the forces of squeezing on downside and upside of contact respectively, sometimes it is necessary to accept $N_1 N_2 = 0$.

Thus for the task shown on Fig.2 we have the next Cauchy problem

$$\begin{array}{r} m\ddot{x} - F_{fr} = 0, \\ y = x - x_0 \sin \omega t, \end{array} \qquad F_{fr} = \begin{cases} -\mu m g \operatorname{sign} \dot{y}, & \text{if} \quad \tau^2 \ddot{y}^2 + \dot{y}^2 \neq 0, \\ f_{st},\ |f_{st}| \leqslant \mu m g & \text{if} \quad \tau^2 \ddot{y}^2 + \dot{y}^2 = 0. \end{cases} \tag{7}$$

To this system initial conditions (3) must be added. The main difficulty of the problem investigation is that it is necessary to look for nonsmooth solutions of (7), (3). For example, the function $\dot{y}(t)$ may be discontinuus. In order to see this fact more clearly let us consider the simple task shown on Fig.1 at $F = 0$. In this case we have the equation

$$m\ddot{x} - F_{fr} = 0 \tag{8}$$

and initial conditions

$$t = 0: \qquad x = 0, \quad \dot{x} = v. \tag{9}$$

If the friction force F_{fr} is determined by expression (1), then we have the unique solution

$$\dot{x} = \begin{cases} v - \mu N t/m, & 0 < t < \tau_{cl} \circeq mv/\mu N, \\ 0, & \tau_{cl} \leqslant t. \end{cases} \tag{10}$$

This is the classical solution. If the friction force F_{fr} is defined by (4) or (5), then we have two solutions

$$a)\ \dot{x}_1 = \begin{cases} v - \mu N t/m, & 0 < t < \tau_{cl}, \\ 0, & \tau_{cl} \leqslant t. \end{cases}, \quad b)\ \dot{x}_2 = \begin{cases} v, & t = 0, \\ 0 & t > 0 \end{cases} \tag{11}$$

The second solution in (11) is an exact solution of the task (8), (9) and (5). However, it is discontinuus solution. Because of this it was ignored by the most of researches. F. Klein was the first who had pointed out the importance of the discontinuus solution in order to avoid the Painleve paradoxes [6]. N.V. Butenin [16] had used this discontinuus solution in order to solve the number of tasks. The physical meaning of discontinuus solution was shown in [18, 19]. The new reinterpret of solutions 11 will be given below in section 6.

The given above formulations of the Coulomb law of friction are not sufficient in order to apply them formally in any cases. One can say that the correct application of this law requires a thorough insight into the details of considered problem. Without this it is impossible to avoid all difficulties only by means of new experiments or new theoretical considerations.

3 The Painleve-Klein Problem. Conventional Approach

Let us consider the task that was studied by P. Painleve [2] and after that was discussed by F. Klein [6]. The task is shown on Fig.3. Namely in this problem P. Painleve had found at the first time the paradoxical situations. Let us show the way of reasoning by Painleve and Klein. Below the improved and enlarged analysis by Painleve and Klein is given, but the basic results are practically the same.

The equation of motion can be represented in the form

$$M_1\ddot{x} = R + P_1 + S\cos\alpha, \quad M_2\ddot{x} = P_2 - S\cos\alpha, \quad 0 < \alpha < \pi/2 \tag{12}$$

where R is the friction force, S is a longitudinal force in the rod. In this task we have to use the Coulomb law in the form (6), where

$$|N| = |S\sin\alpha| = |S|\sin\alpha = S\varepsilon_2\sin\alpha, \quad \varepsilon_2 = \operatorname{sign} S \tag{13}$$

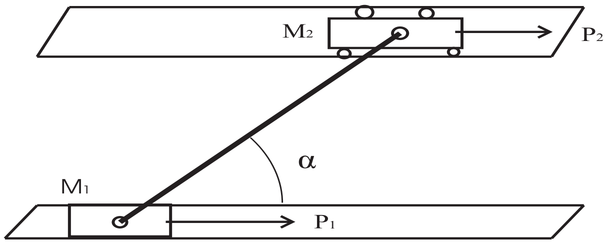

Figure 3: The Painleve-Klein Problem

Thus the friction force R is defined as

$$R = \begin{cases} -\mu S \varepsilon_1 \varepsilon_2 \sin\alpha, & \text{if} \quad \dot{x} \neq 0, \\ f, \ |f| \leqslant \mu |S| \sin\alpha, & \text{if} \quad \dot{x} = 0, \quad \ddot{x} = 0 \end{cases} \tag{14}$$

where $\varepsilon_1 = \operatorname{sign} \dot{x}$. The initial conditions have a form

$$t = 0: \qquad x = 0, \quad \dot{x} = v, \tag{15}$$

where v is an initial velocity.

Let us suppose that at $t > 0$ the masses M_1 and M_2 are not moving: $\dot{x} = 0$. Then instead of equations of motion (12) we have the equation of statics

$$f + P_1 + S\cos\alpha = 0, \quad P_2 - S\cos\alpha = 0. \tag{16}$$

From (16) it follows

$$f = -(P_1 + P_2), \qquad S = P_2/\cos\alpha, \quad |P_1 + P_2| \leqslant \mu |P_2| \tan\alpha \tag{17}$$

The last inequality determines the domain on the plane (P_1, P_2) where the statical solution exists. Thus from the theoretical point of view the static solution exists always, when inequality (17) holds. If $P_2 = 0$, then the system can be at rest only when $P_1 = 0$. But the initial velocity may be different from zero! If $P_1 = 0$, then the statical solution is possible at $v \neq 0$ only if $\mu \tan\alpha \geqslant 1$.

Let us suppose that at $t > 0$ the system is moving with $\dot{x} = \text{const} \neq 0$, $\ddot{x} = 0$. Then we have

$$-\mu S \varepsilon_1 \varepsilon_2 \sin\alpha + P_1 + S\cos\alpha = 0, \quad P_2 - S\cos\alpha = 0, \tag{18}$$

This system has solution if and only if

$$\mu \tan\alpha = 2. \tag{19}$$

The solution has a form

$$P_1 = P_2, \qquad \operatorname{sign} P_1 = \operatorname{sign} v.$$

If $\mu \tan\alpha \neq 2$, then the case $\dot{x} = \text{const} \neq 0$ is impossible.

Let us suppose that at $t > 0$ the masses M_1 and M_2 are moving: $\dot{x} \neq \text{const}$, $\ddot{x} \neq 0$. Then the force S is determined by the expression

$$S = \frac{\gamma P_2 - P_1}{(1+\gamma)\cos\alpha - \mu\varepsilon_1\varepsilon_2\sin\alpha}, \qquad \gamma = \frac{M_1}{M_2}. \tag{20}$$

In the Painleve-Klein analysis the restriction

$$\gamma = 1, \qquad P_2 = 0 \tag{21}$$

were accepted. Multiplying expression (20) by ε_2 and taking into account the equality $|S| = S\varepsilon_2$ we obtain

$$\frac{\varepsilon_2\,(\gamma P_2 - P_1)}{(1+\gamma)\cos\alpha - \mu\varepsilon_1\varepsilon_2\sin\alpha} > 0. \tag{22}$$

For small $\mu > 0$ we have

$$(1+\gamma)\cos\alpha - \mu\sin\alpha > 0, \quad \Rightarrow \quad \mu\tan\alpha < 1+\gamma \tag{23}$$

Then from (22) it follows

$$\varepsilon_2 = \text{sign}\,(\gamma P_2 - P_1)\,. \tag{24}$$

Thus for small μ we have two solutions: one is given by (17) and another is determined by

$$S = \frac{\gamma P_2 - P_1}{d}, \qquad \begin{array}{r} R = -\mu\,\text{sign}\,\nu\,|S|\sin\alpha, \\ d = (1+\gamma)\cos\alpha - \mu\,\text{sign}\,\nu\,\text{sign}(\gamma P_2 - P_1)\sin\alpha. \end{array} \tag{25}$$

This case was not considered by Painleve, since from the Painleve point of view in this case there is no problem. As we see, it is not so. At the moment we have no reasons in order to choose one of two possible solution. However, the Coulomb law of friction is not responsible for such a situation. In fact, our model is not adequate to the reality in many important aspects. In the next section we show why it is important and how to solve the problem of choice.

Let us suppose that the inequality

$$(1+\gamma)\cos\alpha - \mu\sin\alpha < 0 \tag{26}$$

is valid. This inequlity determines a domain of paradoxes accordingly Painleve. Here we have to consider the different cases.

1. The case when $\gamma P_2 - P_1 = 0$.

In such a case the friction is absent and there is nothing to discuss.

2. The case when $\varepsilon_1 = \text{sign}\,\dot{x} = \text{sign}\,\nu = 1$ and $\gamma P_2 - P_1 < 0$.

Then the inequality (22) may be rewritten as

$$\frac{\varepsilon_2\,(\gamma P_2 - P_1)}{(1+\gamma)\cos\alpha - \mu\varepsilon_2\sin\alpha} > 0. \tag{27}$$

In such a case we have two different solutions

$$S = \frac{\gamma P_2 - P_1}{(1+\gamma)\cos\alpha + \mu\sin\alpha}, \qquad \varepsilon_2 = -1 \tag{28}$$

and

$$S = \frac{\gamma P_2 - P_1}{(1+\gamma)\cos\alpha - \mu\sin\alpha}, \qquad \varepsilon_2 = +1. \tag{29}$$

Thus in this case we have three different solutions: (17), (28) and (29).

3. The case when $\varepsilon_1 = 1, \quad \gamma P_2 - P_1 > 0$.

In such a case inequality (27) is not valid for any value of ε_2. Thus in this case we have the unique solution (17). The system is instantly stopping.

4. The case when $\varepsilon_1 = -1, \quad \gamma P_2 - P_1 < 0$.

Then inequality (22) takes a form

$$\frac{\varepsilon_2\,(\gamma P_2 - P_1)}{(1+\gamma)\cos\alpha + \mu\varepsilon_2\sin\alpha} > 0. \tag{30}$$

There is no value of ε_2 to satisfy this inequality. We have the unique solution (17).

5. The case when $\varepsilon_1 = -1, \quad \gamma P_2 - P_1 > 0$. In this case we have two solutions for S

$$S = \frac{\gamma P_2 - P_1}{(1+\gamma)\cos\alpha + \mu\sin\alpha}, \qquad \varepsilon_2 = +1 \tag{31}$$

and

$$S = \frac{\gamma P_2 - P_1}{(1+\gamma)\cos\alpha - \mu\sin\alpha}, \qquad \varepsilon_2 = -1 \tag{32}$$

Again we have three solutions (17), (31), (32).

Painleve presumed that the Coulomb law of friction is responsible for such unsatisfactory situation. Much later the Painleve analysis was confirmed by P. Appell [15]: "it is not necessary to think, that only in exclusive cases there can be possible such difficulties. On the contrary, they arise in the most common cases, at least, at enough large value of factor of friction μ. Because of this new experiments for a finding of the laws of friction, which is not resulting more in these difficulties, are necessary". However, L. Prandtl [14] and F. Klein [6] were not agree with the conclusions by Painleve. F. Klein had pointed out that if $\nu < 0$, then there is a unique solution $\dot{x} = 0$. In order to avoid contradictions in the case $\nu > 0$, F. Klein offer to accept the statical solution $\dot{x} = 0$ as well. But F. Klein did not explain why we have to do this. The Painleve considerations are supposed to be right even in modern books — see, for example, [17].

4 The Painleve-Klein problem. Alternative approach

It is not difficult to see that the Painleve analysis, shown in previous section, is needed in some additions and improvements, especially from the physical point of view, since the task shown on Fig.3 is undefined in some important aspects. There is useful heuristic principle in mechanics: if one accepts some assumptions, then it is necessary to show the model in which these assumptions may be exactly realised. Only in such a case one can be shure that accepted assuptions have physical meaning. In the Painleve-Klein problem the model is described by equations (12) and (14). From the pure mathematical point of

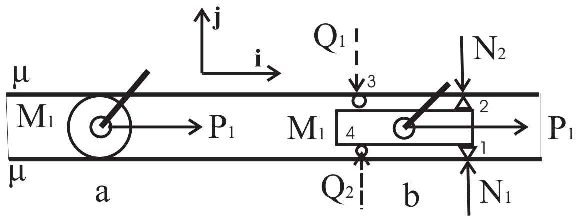

Figure 4: The structure of the downslipper

view this model is certain mathematical object and we have to study its properties. What these properties are, we can not speak about paradoxes, for the mathematical object can have the most fantastic properties. From the physical point of view the situation varies. We speak about paradoxes when the results of the decision of this or that task contradict common sense. But in such a case the physical statement of the task itself should not contradict common sense. Let's discuss the model discribed by equations (12) and (14). Equations (12) show that there are no moments on the ends of rod. This means that the masses M_1 and M_2 may be rotated with respect to the rod fluently. It is easy to see that equations (12) and (14) correspond to the case shown on Fig.4a. Let us underline that the mass M_1 is touching either the upside or downside of the gap. From Fig.4a it is seen that in considered case the Coulomb law of friction can't be used since we have rolling motion of M_1 instead of the sliding, which is possible only if we exclude the turn of the body M_1 with respect to the rod. However, in such a case the force in the rod will not be a longitudinal force any more and equations (12) must be changed. Thus the statement of the problem considered in the previous section is physically meaningless and the Coulomb law of friction is not responsible for the paradoxes. More realistic structure of the slipper is shown on Fig.4b. The forces acting on the slipper M_1 from the foundation are shown on Fig.4b. Some other cases are shown on Fig.5a-e. The difference between cases Fig.5a and Fig 5b is that in the second case the slipper can't rotate with respect to the rod. Below we deduce the equations for the case on Fig.4b. Let us write down the equations of motion

$$M_1 \ddot{x}\mathbf{i} = P_1 \mathbf{i} + \mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3 + \mathbf{R}_4 + S(\cos\alpha \mathbf{i} + \sin\alpha \mathbf{j}), \tag{33}$$

$$M_2 \ddot{x}\mathbf{i} = P_2 \mathbf{i} - S(\cos\alpha \mathbf{i} + \sin\alpha \mathbf{j}) + R_5 \mathbf{j}, \tag{34}$$

where $R_5 j$, R_k, $(k = 1, 2, 3, 4)$ are the reactions acting on the body M_2 and points $1, 2, 3, 4$ of the slipper M_1 (see Fig.4b) respectively, S is a longitudenal force in the rod, $S > 0$ when the rod is stretched. For the reactions R_k we have

$$\mathbf{R}_1 = R_1 \mathbf{i} + N_1 \mathbf{j}, \quad \mathbf{R}_2 = R_2 \mathbf{i} - N_2 \mathbf{j}, \quad \mathbf{R}_3 = -Q_1 \mathbf{j}, \quad \mathbf{R}_4 = Q_2 \mathbf{j} \tag{35}$$

and the restrictions

$$N_1 \geqslant 0, \quad N_2 \geqslant 0, \quad N_1 N_2 = 0, \quad Q_1 \geqslant 0, \quad Q_2 \geqslant 0, \quad Q_1 Q_2 = 0 \tag{36}$$

are valid. Besides, the functions R_1, R_2 are defined by the Coulomb law of friction. From equations (33)–(35) it follows

$$M_1\ddot{x} = P_1 + R_1 + R_2 + S\cos\alpha, \qquad M_2\ddot{x} = P_2 - S\cos\alpha, \tag{37}$$

$$N_1 - N_2 - Q_1 + Q_2 + S\sin\alpha = 0, \qquad R_5 - S\sin\alpha = 0,$$
$$N_1N_2 = 0, \quad Q_1Q_2 = 0. \tag{38}$$

The Coulomb law of friction can be written in the form

$$R_1 = \begin{cases} -\mu N_1 \operatorname{sign}\dot{x}, & \text{if} \quad \dot{x} \neq 0,\ \ddot{x} \neq 0, \\ f_1,\ |f_1| \leqslant \mu N_1, & \text{if} \quad \dot{x} = 0,\ \ddot{x} = 0, \end{cases} \tag{39}$$

$$R_2 = \begin{cases} -\mu N_2 \operatorname{sign}\dot{x}, & \text{if} \quad \dot{x} \neq 0,\ \ddot{x} \neq 0, \\ f_2,\ |f_2| \leqslant \mu N_2, & \text{if} \quad \dot{x} = 0,\ \ddot{x} = 0. \end{cases} \tag{40}$$

We need to know the sum

$$R_1 + R_2 = \begin{cases} -\mu\,(N_1 + N_2) \operatorname{sign}\dot{x}, & \text{if} \quad \dot{x} \neq 0,\ \ddot{x} \neq 0, \\ f_1 + f_2,\ |f_1| \leqslant \mu N_1,\ |f_2| \leqslant \mu N_2, & \text{if} \quad \dot{x} = 0,\ \ddot{x} = 0, \end{cases} \tag{41}$$

where the restrictions (36) must be taking into account. Now we are able to compare the statements (12), (14) and (37), (38), (41). If $Q_1 = Q_2 = 0$, then both statements are the same. Let us pay attention that the Coulomb law of friction is applying in both statements in the same manner. If $Q_1 \neq 0$ or $Q_2 \neq 0$, then these statements are different very much. First of all, the system (36)–(41) with initial conditions

$$t = 0: \qquad x = 0, \quad \dot{x} = v \tag{42}$$

is incomplete one. We need one more equation. There exist different ways. The most reliable way is to take into account the elasticity of the gap walls. Strictly speaking in such a case we must not only add a new equation but replace the first equation of system (38) by the next equation

$$M_1\ddot{y} = N_1 - N_2 - Q_1 + Q_2 + S\sin\alpha, \tag{43}$$

where y is vertical coordinate of the mass center of M_1. Let us suppose that the slipper M_1 can be rotated by the small angle φ. In such a case the vertical coordinates of the points $1, 2, 3, 4$ may be found as

$$y_1 = y + l_1\varphi, \quad y_2 = y + l_1\varphi, \quad y_3 = y - l_1\varphi, \quad y_4 = y - l_1\varphi. \tag{44}$$

For the reactions N_1, N_2, Q_1, Q_2 the next constitutive equations may be accepted

$$N_1 = -c\,[1 - \theta(y + l_1\varphi)]\,(y + l_1\varphi), \quad N_2 = c\theta(y + l_1\varphi)(y + l_1\varphi),$$

$$Q_1 = c\theta(y - l_1\varphi)(y - l_1\varphi), \quad Q_2 = -c\,[1 - \theta(y - l_1\varphi)]\,(y - l_1\varphi), \tag{45}$$

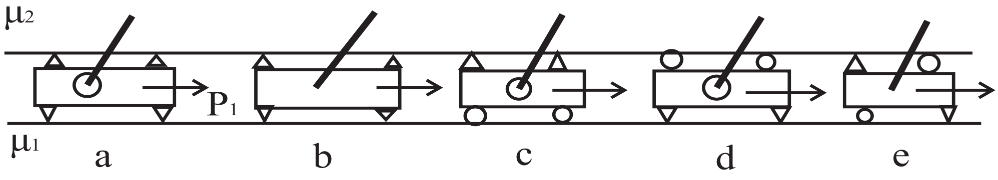

Figure 5: The slipper in the gap

where l_1 is a parameter of the length dimension, $c > 0$ is a stiffness of elastic foundation, and $\theta(z)$ is the characteristic function of the domain $z \geqslant 0$

$$\theta(z) = \begin{cases} 1, & \text{if} \quad z \geqslant 0, \\ 0, & \text{if} \quad z < 0. \end{cases}, \quad \begin{array}{l} \theta_+ \equiv \theta(y + l_1\varphi), \\ \theta_- \equiv \theta(y - l_1\varphi). \end{array} \tag{46}$$

The additional equation can be accepted in the next form

$$M_1 r^2 \ddot{\varphi} = l_2\,(N_1 - N_2 + Q_1 - Q_2) + l_3(R_1 - R_2) - \varepsilon P_1, \tag{47}$$

where l_2, l_3, ε are the parameters of the length dimension, r is an inertia radius of the slipper. We obtain the closed system of equations (37), (39), (40), (43), (45), (47). To this system we have to add the initial conditions which can be taking, for example, in the next fom

$$t = 0: \qquad x = 0, \quad \dot{x} = \nu, \quad y = \dot{y} = \varphi = \dot{\varphi} = 0. \tag{48}$$

Only now we have the well-defined task from the physical point of view. The final statement can be represented as

$$M_1\ddot{x} = P_1 + R_1 + R_2 + S\cos\alpha, \qquad M_2\ddot{x} = P_2 - S\cos\alpha,$$

$$M_1\ddot{y} + 2cy = S\sin\alpha, \quad M_1 r^2\ddot{\varphi} + 2cl_2l_1\varphi = l_3(R_1 - R_2) - \varepsilon P_1, \tag{49}$$

where

$$R_1 + R_2 = \begin{cases} -\mu c\,|y + l_1\varphi|\,\mathrm{sign}\,\dot{x}, & \text{if} \quad \dot{x} \neq 0,\ \ddot{x} \neq 0, \\ f_1 + f_2, & \text{if} \quad \dot{x} = 0,\ \ddot{x} = 0, \end{cases} \tag{50}$$

where

$$|f_1| \leqslant \mu c(1 - \theta_+)\,(y + l_1\varphi)\,, \qquad |f_2| \leqslant \mu c\theta_+(y + l_1\varphi). \tag{51}$$

$$R_1 - R_2 = \begin{cases} \mu c(y + l_1\varphi)\,\mathrm{sign}\,\dot{x}, & \text{if} \quad \dot{x} \neq 0,\ \ddot{x} \neq 0, \\ f_1 - f_2, & \text{if} \quad \dot{x} = 0,\ \ddot{x} = 0. \end{cases} \tag{52}$$

The Cauchy problem (49), (50), (52), (48) is a physically correct statement of the Painleve-Klein task. If we doubt in the Coulomb law of friction, then we have to show that the Cauchy problem is not well-defined. However, it is not so. Let us transform system (49)–(52). For this we accept the restriction $r^2 = l_1 l_2$. In such a case from (49)–(52) one can derive the equations

$$(M_1 + M_2)\ddot{x} = P_1 + P_2 + R_1 + R_2,$$

$$M_1\ddot{y} + 2cy = (\frac{\gamma}{1+\gamma}P_2 - \frac{1}{1+\gamma}P_1)\tan\alpha - \frac{\tan\alpha}{1+\gamma}(R_1 + R_2),$$
$$M_1\ddot{z} + 2cz = \frac{l_3}{l_2}(R_1 - R_2) - \frac{\varepsilon}{l_2}P_1, \quad z = y + l_1\varphi. \tag{53}$$

The force S may be found from the equation

$$S = (P_2 - M_2\ddot{x})/\cos\alpha. \tag{54}$$

Let's note that the friction forces $R_1 + R_2$, $R_1 - R_2$ are expressed in terms of the variable z by means of (50) and (52). Initial conditions for system (53) has a form

$$t = 0: \qquad x = 0, \quad \dot{x} = \nu, \quad y = z = \dot{y} = \dot{z} = 0. \tag{55}$$

For small $t > 0$ the system is moving. So making use (50) and (52) we can transform system (53) to the next form

$$(M_1 + M_2)\ddot{x} = -\mu\varepsilon_1 c\,|z| + P_1 + P_2, \quad \varepsilon_1 \equiv \operatorname{sign}\dot{x} = \operatorname{sign}\nu,$$
$$M_1\ddot{y} + 2cy = \mu\varepsilon_1 c\,|z|\,\frac{\tan\alpha}{1+\gamma} + (\frac{\gamma}{1+\gamma}P_2 - \frac{1}{1+\gamma}P_1)\tan\alpha,$$
$$M_1\ddot{z} + 2c(1 - \frac{\mu\varepsilon_1}{2}\frac{l_3}{l_2})z = -\frac{\varepsilon}{l_2}P_1, \tag{56}$$

The Cauchy problem (56)–(55) is well-defined and obviously has unique solution. Of course, we must have in mind that this problem has a meaning only when

$$\begin{aligned} 0 \leqslant \dot{x} \leqslant \nu, \quad \text{if} \quad \varepsilon_1 = +1, \\ \nu \leqslant \dot{x} \leqslant 0, \quad \text{if} \quad \varepsilon_1 = -1. \end{aligned} \tag{57}$$

We see that there are no problems when the Coulomb law of friction is using in the Painleve-Klein task. While we were forced to take into account an elasticity of the gap walls, nevertheless this was not connected with the law of friction but due to physical requirements only.

If we wish to use the rigid body model, then we have to make the passage to the limit $c \to \infty$. In such a case we obtain the rigid body model. If $c \to \infty$, then $z \to 0$ and $y \to 0$, but

$$\lim_{c\to\infty} cy = T \neq 0, \qquad \lim_{c\to\infty} cz = Z \neq 0. \tag{58}$$

Instead of the Cauchy problem (53) we shall get

$$(M_1 + M_2)\ddot{x} = P_1 + P_2 + R_1 + R_2, \quad 2Z = \frac{l_3}{l_2}(R_1 - R_2) - \frac{\varepsilon}{l_2}P_1,$$
$$2T = (\frac{\gamma}{1+\gamma}P_2 - \frac{1}{1+\gamma}P_1)\tan\alpha - \frac{\tan\alpha}{1+\gamma}(R_1 + R_2). \tag{59}$$

Equalities (50) and (52) take a form

$$R_1 + R_2 = \begin{cases} -\mu\,|Z|\operatorname{sign}\dot{x}, & \text{if} \quad \dot{x} \neq 0,\ \ddot{x} \neq 0, \\ f_1 + f_2, & \text{if} \quad \dot{x} = 0,\ \ddot{x} = 0, \end{cases}$$

$$R_1 - R_2 = \begin{cases} \mu Z \operatorname{sign} \dot{x}, & \text{if} \quad \dot{x} \neq 0,\ \ddot{x} \neq 0, \\ f_1 - f_2, & \text{if} \quad \dot{x} = 0,\ \ddot{x} = 0, \end{cases} \tag{60}$$

where

$$|f_1| \leqslant \mu(1-\eta_1)\,|Z|\,, \qquad |f_2| \leqslant \mu\eta_1\,|Z|\,. \tag{61}$$

In (60) and (61) the next notation

$$\eta_1 = \lim_{z\to 0}\theta_+ = \begin{cases} 1, & \text{if} \quad z \to +0, \\ 0, & \text{if} \quad z \to -0, \end{cases}$$

$$\eta_2 = \lim_{y - l_1\varphi\to 0}\theta_- = \begin{cases} 1, & \text{if} \quad y - l_1\varphi \to +0, \\ 0, & \text{if} \quad y - l_1\varphi \to -0 \end{cases} \tag{62}$$

is used. As initial conditions we have to accept equalities (42).

Now we are able to compare the statements (12)–(15) and (59)–(62), (42). We see that they are quite different, while both of them are using the model of rigid body and standard form of the Coulomb law of friction. Let's consider the solution of system (59)–(62), (42). The Coulomb law has different forms for the motion and for the rest. So we have to consider these cases separately.

Let's suppose that $\dot{x} = 0,\ \ddot{x} = 0$. Then we get

$$P_1 + P_2 + f_1 + f_2 = 0, \quad 2T = (\frac{\gamma}{1+\gamma}P_2 - \frac{1}{1+\gamma}P_1)\tan\alpha - \frac{\tan\alpha}{1+\gamma}(f_1+f_2),$$

$$2Z = \frac{l_3}{l_2}(f_1 - f_2) - \frac{\varepsilon}{l_2}P_1, \quad |f_1| \leqslant \mu(1-\eta_1)\,|Z|\,, \qquad |f_2| \leqslant \mu\eta_1\,|Z|\,. \tag{63}$$

Now we have to consider two cases

$$a) \quad \eta_1 = 1 \quad \Rightarrow \quad f_1 = 0, \quad |f_2| \leqslant \mu\,|Z|\,, \quad Z \geqslant 0 \tag{64}$$

and

$$b) \quad \eta_1 = 0 \quad \Rightarrow \quad |f_1| \leqslant \mu\,|Z|\,, \quad f_2 = 0, \quad Z \leqslant 0 \tag{65}$$

In case (64) we have

$$f_2 = -(P_1 + P_2),\ 2T = P_2\tan\alpha,\ 2Z = \frac{l_3 - \varepsilon}{l_2}P_1 + \frac{l_3}{l_2}P_2 \geqslant 0, \tag{66}$$

$$|P_1 + P_2| \leqslant \frac{\mu}{2}\left|\frac{l_3-\varepsilon}{l_2}P_1 + \frac{l_3}{l_2}P_2\right|. \tag{67}$$

Inequality (67) determines the domain of the static solution existence under $\eta_1 = 1$.

In case (65) we get

$$f_1 = -(P_1 + P_2),\ 2T = P_2\tan\alpha,\ 2Z = -\frac{l_3 + \varepsilon}{l_2}P_1 - \frac{l_3}{l_2}P_2 \leqslant 0, \tag{68}$$

$$|P_1 + P_2| \leqslant \frac{\mu}{2l_2}\,|(l_3+\varepsilon)P_1 + l_3P_2|\,. \tag{69}$$

It is necessary to have in mind that the existence of two cases (64) and (65) does not mean that there is non-uniqueness of solutuion. These cases correspond to different physical conditions. Let P_2 be absent as in the Painleve-Klein problem. In such a case the statical solution exist only when $P_1 > 0$ and

$$a)\quad \frac{\mu}{2}\frac{l_3-\varepsilon}{l_2} \geqslant 1, \qquad b)\quad \frac{\mu}{2}\frac{l_3+\varepsilon}{l_2} \geqslant 1 \tag{70}$$

Let's the case when $\dot{x} = v = \text{const}$ and $\ddot{x} = 0$. System (59)–(61) takes a form

$$-\mu\varepsilon_1 |Z| + P_1 + P_2 = 0, \quad 2T = \mu\varepsilon_1 |Z| \frac{\tan\alpha}{1+\gamma} + \left(\frac{\gamma}{1+\gamma}P_2 - \frac{1}{1+\gamma}P_1\right)\tan\alpha,$$

$$(2 - \mu\varepsilon_1\frac{l_3}{l_2})Z = -\frac{\varepsilon}{l_2}P_1. \tag{71}$$

This system has a solution only when

$$P_2 = \frac{\mu\varepsilon_1 |\varepsilon P_1|}{|2l_2 - \mu\varepsilon_1 l_3|} - P_1. \tag{72}$$

At last, let's consider the case $\ddot{x} \neq 0$. Equation (59) take a form

$$(M_1 + M_2)\ddot{x} = -\mu\varepsilon_1 |Z| + P_1 + P_2, \quad \varepsilon_1 \equiv \operatorname{sign}\dot{x} = \operatorname{sign} v,$$

$$2T = (\mu\varepsilon_1 |Z| + \gamma P_2 - P_1)\frac{\tan\alpha}{1+\gamma}, \quad (2 - \mu\varepsilon_1\frac{l_3}{l_2})Z = -\frac{\varepsilon}{l_2}P_1. \tag{73}$$

It is easy to see that this system has a unique solution in all cases and paradoxes of any kind are absent. This means that paradoxes shown in previous section are result of unsatisfactory statement of task but not due to the Coulomb law of friction. However, we have to underline that absence of paradoxes does not mean that the task (73) is good from physical point of view. It is not so. As a matter of fact when making a limit passage $c \to \infty$ we have lost a number of important properties of considered system. For example, the Cauchy problem has a meaning not only for initial condition (55). It is possible to put, for example, the next condition at $t = 0$

$$x = 0, \quad \dot{x} = v, \quad y = a \neq 0, \quad z = 0, \quad \dot{y} = \dot{z} = 0. \tag{74}$$

In this case the limit passage $c \to \infty$ leads to contradiction since

$$\lim_{c\to\infty} cy \to \infty,$$

what is natural from physical point of view. Besides, system (56) shows that if

$$\frac{\mu\varepsilon_1}{2}\frac{l_3}{l_2} > 1, \tag{75}$$

then we have almost instant shut-down of the system. We can't see it from equation (73). So from practical point of view it is much better to use equations (56).

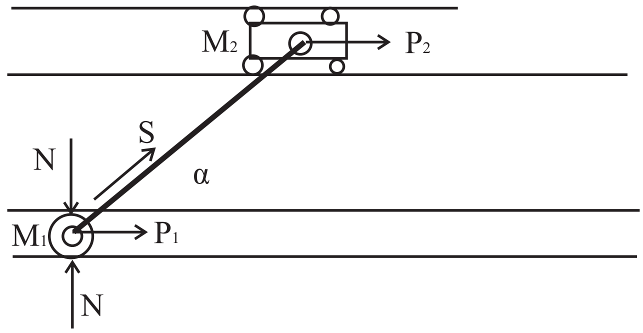

Figure 6: The modified task by Painleve

5 The modified Painleve-Klein problem

We saw that the classical task by Painleve, shown on Fig.3 and Fig.4a, has no physical meaning. Because of this we used the improved version shown on Fig.4b. There is another possibility to improve the statement by Painleve — see Fig.6.

6 The interpretation of instant shut-down of the body with finite mass

Let's turn back to the simplest task considered in section 2. Let's consider the body moving due to inertia along the rough surface — see Fig.1, where $F = 0$. We saw that in this task there are two solutions (11). At the first time an importance of this fact was marked by F. Klein. However the most of scientists do not accept the Klein result. It is easy to understand the main reason of this. In the Klein proposition we deal with the instantaneous stopping of a body with the finite mass. Everybody knows that in such a case the infinitely big force is needed what is impossible in reality. In paper [18] it was shown the physical sense of discontinuous solution and how to choose the necessary solution from two possible solutions. Nevertheless, as it is became clear from discussions with other peoples, there exists the necessity to consider the Klein proposition more carefully.

Let us consider the body moving due to an inertia along the rough surface. A friction is determined by the Coulomb law. Using notation shown in Fig.1 we can write the next system of equations

$$m\ddot{y} = F, \quad F = \begin{cases} -\mu m g \dot{x}/|\dot{x}|, & \text{if } \dot{x} \neq 0 \\ f_{st}, \; |f_{st}| \leqslant \mu m g, & \text{if } \dot{x} = 0 \end{cases} \tag{76}$$

where y is a position of the mass center C, x is a position of a point on the contact surface, m is the body mass, g is the gravity acceleration. Initial conditions can be chosen in the form

$$t = 0: \quad x = y = 0, \quad \dot{x} = \dot{y} = v \tag{77}$$

System (76) is theoretically exact. However in order to solve system (76)–(77) we must accept additional conventions that can be taken in different forms. The conventional way is to accept the model of rigid body. In such a case we have that $z = x = y$ and instead of system (76) we shall get the system

$$m\ddot{z} = F, \qquad F = \begin{cases} -\mu m g \dot{z}/|\dot{z}|, & \text{if } \dot{z} \neq 0, \\ f_{st}, \ |f_{st}| \leqslant \mu m g, & \text{if } \dot{z} = 0, \end{cases} \tag{78}$$

that can be solved without any difficulties. It is easy to see that problem (77)–(78) has two solutions

$$a) \quad \dot{z}_1 = \begin{cases} \nu - \mu g t, & \text{if } t < \nu/\mu g, \\ 0, & \text{if } t > \nu/\mu g \end{cases} \qquad \text{and} \qquad b) \quad \dot{z}_2 = \begin{cases} \nu, & \text{if } t = 0, \\ 0, & \text{if } t > 0. \end{cases} \tag{79}$$

In the first of these solutions the acceleration has the discontinuity of the finite magnitude. In the second solution the acceleration has the discontinuity of the infinite magnitude that is considered to be impossible for the real body.

Now we have arrived to the point of discordance of opinions. First of all, we must understand the meaning of the function $z(t)$ in system (78) or in solutions (79). If $z(t)$ is a position of the mass center, then $z(t)$ is certainly determined by the first solution from expressions (79) whereas the second solution has no physical sense. Thus if we are able to prove that the function $z(t)$ in problem (77)–(78) has the only sense of the position of the mass center, then the Klein solution $b)$ from (6.4) must be eliminated. Is it possible to prove this presumption? Note that problem (77)–(78) considered from the mathematical point of view does not know our conceptualization of the meaning of function $z(t)$. Actually, in order to obtain system (78) from system (76) only the relation $x = y$ is important. It does not matter what kind of word explanations we shall use. This means that it is impossible to find the meaning of $z(t)$ in system (78) from formal considerations without additional investigations. From the physical point of view it is clear that the interpretations of functions $z(t)$ for solutions $a)$ and $b)$ in (79) must be different. For example, if $z(t)$ is the position of contact surface, then solution $b)$ in (79) can be realized in reality since the contact surface has no mass while the center of mass may keep its motion. Anyhow, in order to find the detailed answer we must investigate the problem more carefully.

6.1 Enlarged model

From equations (76) we see that a suitable model must have at least two degrees of freedom. The simplest model of such a kind can be taken in the form shown in Fig.7. The system consist of the rigid framework of the mass m with a rigid horizontal rod inside and a body of mass M that restrained to move along the rod. The spring of stiffness c connects the body M with the framework m. The latter simulates a contact surface whereas the body M simulates the centre of mass. The enlarged model tends to the rigid body model as $c \to \infty$. It is not difficult to derive equations of motion for this model. They have the form

$$\ddot{x} + \omega_0^2 (x - y) = F/m, \qquad \ddot{y} + \omega^2 (y - x) = 0 \tag{80}$$

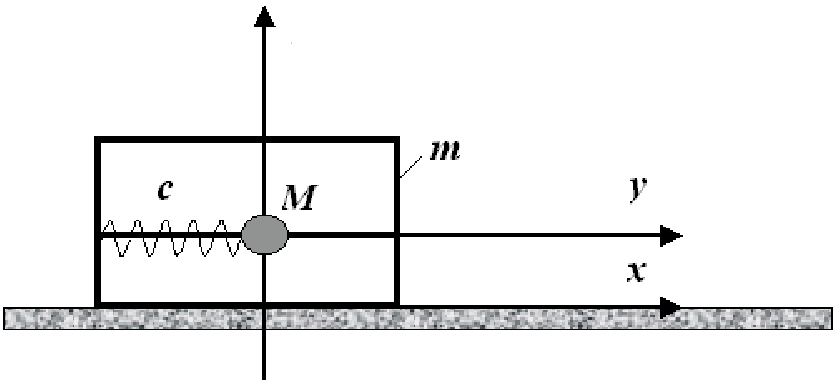

Figure 7: The enlarged model

where the friction force F is defined by expression (76), $\omega_0^2 = c/m$, $\omega^2 = c/M$, x and y determine the positions of the framework and the body M respectively. The initial conditions retain prior form (77). If $m \neq 0$, then for small $t > 0$ the solution of problem (80), (77) has the form

$$x = vt - \frac{1}{2}\mu g t^2 - \frac{M}{m}\frac{\mu g}{\Omega^2}\left(1 - \cos\Omega t\right), \tag{81}$$

$$y = vt - \frac{1}{2}\mu g t^2 + \frac{\mu g}{\Omega^2}\left(1 - \cos\Omega t\right), \tag{82}$$

where

$$\Omega^2 = \omega_0^2 + \omega^2 = c\left(\frac{1}{m} + \frac{1}{M}\right). \tag{83}$$

Solution (81), (82) is valid for such t that

$$\dot{x} = v - \mu g t - \frac{M}{m}\frac{\mu g}{\Omega}\sin\Omega t > 0. \tag{84}$$

The moment of stopping $t = \tau$ must be found from the equation

$$v - \mu g\tau - \frac{M}{m}\frac{\mu g}{\Omega}\sin\Omega\tau = 0. \tag{85}$$

In what follows we shall assume that $m/M \ll 1$. Now we must consider solution (81)–(82) more carefully.

6.2 The model of rigid body

It is obvious that the enlarged model tends to the rigid body model as $c \to \infty$. This means that $\Omega \to \infty$. In such a case from expressions (81)–(82) it follows

$$x = y = vt - \frac{1}{2}\mu g t^2, \qquad t < \tau_{cl} \equiv \frac{v}{\mu g}, \tag{86}$$

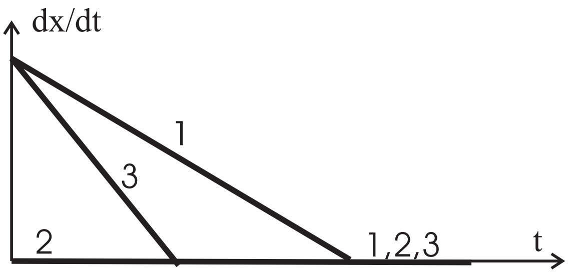

Figure 8: line 1 is the classical solution; line 2 is the solution for the massless framework; line 3 is the velocity of framework when m<<M

where τ_{cl} is the classical time of stopping. This solution is shown in Fig.8. Thus we see that solution z_1 from (79) is the limiting case

$$z_1 = \lim_{c\to\infty} x\,(t,c,m) = \lim_{c\to\infty} y\,(t,c,m) = \nu t - \frac{1}{2}\mu g t^2.$$

In this case the meaning of the solution z_1 is obvious.

6.3 The enlarged model with the massless framework

Let us consider the case such that

$$m \ll M$$

If $m \to 0$, then $\Omega \to \infty$ and $m\Omega^2 \to c$. The time of stopping τ_* must be found from equation (85). Let us suppose that $\Omega\tau_* \ll 1$. In such a case the approximate solution of equation (85) has the form

$$\tau_* = \frac{m}{m+M}\frac{\nu}{\mu g} \simeq \frac{m}{M}\tau_{cl} \ll \tau_{cl}. \tag{87}$$

We see that the strong inequality

$$\Omega\tau_* = \sqrt{\frac{m}{M}}\sqrt{\frac{c}{M}}\frac{\nu}{\mu g} \ll 1$$

is valid if $m \ll M$. For small intervals of time $t < \tau_*$ solution (81) and (82) can be rewritten in the next form

$$x = \nu t - \frac{1}{2}\left(1 + \frac{M}{m}\right)\mu g t^2, \qquad y = \nu t. \tag{88}$$

Now we can see the sense of solution z_2

$$\dot{z}_2 = \lim_{m\to 0} \dot{x}\,(t,c,m) = \begin{cases} \nu, & \text{if } t = 0, \\ 0, & \text{if } t > 0. \end{cases} \tag{89}$$

Here the function z_2 can be only treated as the position of the framework but not the position of the center of mass. The motion of the latter in case (89) is defined by the expression

$$y = \frac{v}{\omega} \sin \omega t, \qquad \omega^2 = \frac{c}{M}, \qquad t \geqslant 0. \tag{90}$$

This expression is valid if the force in the spring is less than $\mu M g$

$$|cy(t)|_{\max} \leqslant \mu (m + M) g \approx \mu M g. \tag{91}$$

This condition will be satisfied for all times if the initial velocity satisfies the inequality

$$|v| \leqslant \mu g \sqrt{\frac{M}{c}}, \quad v_c = \mu g \sqrt{\frac{M}{c}}, \tag{92}$$

where v_c is a critical velocity. If this condition does not hold good, then inequality (91) the possibility to find the interval of time when solution (89)–(90) is valid. After that it is necessary to solve the Cauchy problem with new initial conditions.

6.4 Discussion

The second solution in (79) is called the Klein hypothesis. We saw that as a matter of fact it is not a hypothesis but the essential corollary of the conventional statement of all problems with Coulomb friction. The usual objection against the Klein hypothesis must be rejected if we accept the right interpretation for the function $z(t)$ in system (78). For the first solution in (79) it is quite possible to use two different interpretations, i.e. the function $z(t)$ can be considered as the position of the mass center or as the position of the contact surface. However for the second solution in (79) we have the only interpretation. This means that in general case the function $z(t)$ in (78) must be treated as the position of some point on the contact surface. Note that in many cases it is very important to take into account the second solution in (79) if we want to avoid contradictions of different kind.

Above we saw that exact statement of a task about movement of a body on the rough surface leads to unclosed system of equations. This is the direct indication on the singularity of the given task, for a problem of closing in many cases can not be solved by unique manner. In the given work the conventional closing of system (76) is used. As a result we have received the closed system, but in exchange we have got a new problem. Namely, the sense of function $z(t)$, strictly speaking, has remained uncertain. Maybe, the closing of system was made too roughly and rectilinearly, and the task revenges us, throwing up the senseless decision? Or, maybe, the Nature signals us about some important fact, which we should take into account? To the answers to these questions also is devoted the given subsection.

It is clear, that for the answer to the put above questions it is necessary to consider a task in the extended statement including additional the factors. It also was done above. By result of this analysis are two central conclusions.

First: the function $z(t)$ in system characterizes a position of point of a contact surface, but not a position of the center of mass; it at once removes traditional objection against use of the decision with instant shut-down, for instantly (or practically instantly) the body of infinitesimal mass stops.

The second conclusion: two solutions (79) cannot be understood so, that one of them is realized actually. As a matter of fact these solutions give us only top and bottom boundaries, between which a true solution is placed, but true movement of a body can not be found from the system (78).

As well as any theoretical statements given conclusions require experimental check. Let's carry out the following mental experiment. Let's take a bookcase with a number of horizontal shelves for the books. The case should be easy and rigid as much as possible. Besides we shall take a large load that can be placed on one shelf. Now it is possible to begin experiments. Previously we have to notice, that from a point of view of system (78) it has no importance on which shelf will be the load is located. Let's carry out a series of experiments, in each of which we shall use the same initial speed. In the first experiment the load is located on the bottom shelf. Let's measure the distance, gone the bookcase on inertia. If the center of mass of the bookcase with a load will be located enough close to a floor, the case will pass distance close to the predicted by the classical solution. The diagram for speed of a point on a contact surface will be close to the classical solution, i.e. to the line 1 on the Fig. 8. In the second experiment a load is arranged on the second shelf from below. The center of mass of a body thus will appear above, than in the first experiment. The measurements will show that the case will pass smaller distance rather than in the first experiment. The diagram for speed also moves to the left. Repeating these experiments and lifting a load higher and higher, we shall see, that the diagram of speed will nestle on an axis of ordinates, and gone distance became less and less. Certainly, all told carries speculative character. However it is difficult to doubt in told, for such behavior of a body is predicted by the analysis of behavior of the extended model.

Thus, we see, that the solutions (79) really give us only boundaries, in which there is a required solution, but itself the true solution, describing true movement of a body, is determined by the height parameter of the center of mass, which does not contain at all in system (78). From here also arises uncertainty in the solution of this system allowing defining only boundaries, in which there is a solution, but not solution.

The following picture of movement of extended model follows from all told. We admit, that its framework is opaque and has neglectably small mass. Admit that we observe only movement of a framework and do not know about content of this box. Let at $t < 0$ the system moved with constant velocity v under action of external forces, and the spring is considered not deformed. At $t = 0$ action of external forces suddenly stops, and body continues to move at action forces of friction. As the movement of this body will look from the point of view of the external observer? As the framework is considered to be inertialess, it instantly will stop, but the load M will continue the movement stretching a spring. Force of elasticity of a spring, on the one hand slows down movement of a load M, but, on the other hand, it acts on the framework. Two variants further are possible. In the first variant, under action of force of elasticity the load M will stop at some moment of time $t > 0$. It means, that force of elasticity is not bigger than the biggest possible force of friction of rest $\mu M g$. From the point of view of the external observer the body stands on a place, though invisible outside vibrations of a load M proceed inside a framework. The first variant is realized, if the initial speed was less than some critical velocity ν_c, determined by an inequality (92).

The second variant is more interesting. Let initial speed exceeds the critical velocity.

Then the force of elasticity will exceed maximal force of friction of rest at the moment of time t_1, when the load M will continue the movement. At this moment of time the framework will be broken from a place and instantly will catch up a load M, i.e. will restore the initial position with respect to the load M. Let's remind that the speech goes about the inertialess framework and spring. After that the framework will stop this process will be repeated, but already with smaller by initial velocity. The external observer thus will observe strange picture of movement of the framework. After cancellation external forces the framework will stop, some time will stand on a place, then will make the jump and again will stop. The number of such jumps depends on size initial velocity. After the appropriate number of jumps the framework is finally will stop, and the load, invisible to the observer, will make vibrations inside a framework. Basically, something similar can be observed in experiment. In this hypothetical case not only velocity, but also distance will be discontinuous functions of time. Certainly, if to a framework to attribute as much as small weight, the continuity of movement restores. But at very small weight this continuous movement will appear enough close to described above.

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The Main Direction of the Development of Mechanics for XXI Century*

Abstract

The multi-spin continuum mechanics is an extension of the Cosserat continuum (single-spin continuum). The report presents a general theory and its applications to the derivation of the equations by Maxwell.

1 Introduction

Mechanics before Newton had been remaining by a collection of many important, but separated facts. Newton was the first, who set up a problem of construction of mechanics, as a science of the first principles. As the first principles Newton pointed out the three Laws of Motion, but he did not consider them as a sufficient foundation for a general construction of mechanics. For example, in work [1] Newton wrote: "Vis inertia is a passive principle, by means of which bodies stay in their motion or rest, receive motion, proportional applied to them force, and resist so, as far as meet a resistance (this is the statement of all three laws, P. Zh.). Only because of this there could not be a motion in the world. Other principle was necessary to reduce bodies in motion and, since they are in motion, one more principle is required for preservation of motion. For from various additions of two motions it is quite clear, that in the world there is not always the same momentum. If two balls, joint thin rod, rotate round a common center of gravity by uniform movement, while the center is uniformly gone on a direct line conducted in a plane of their circular movement, the sum of motions of two balls in that case, when the balls are on a direct line circumscribed by their center of gravity, will be more, than the sum of their motions, when they are on a line, perpendicular to this direct. From this example it is clear, that the motion can be received and to be lost" — see p.301. These words were written in 1717 and give clear impression about a level of development of mechanics in the first quarter of the XVIII century. The programmed Newton's idea about construction of mechanics on the base of the first principles had played a huge stimulating role. Euler carried out the realization of this program. In period with 1732 on 1755 Euler has developed the concept, which now is accepted as

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Newtonian mechanics. In this concept the translation of mechanics on language of the differential equations was made. The stage of the construction of Newtonian mechanics in the fundamental plan was finished by the memoir of Euler[2] "Discovery of a new principle of mechanics", published in 1752. In that time Euler was considering, that the principle, opened by him, is possible to consider "as a unique base of mechanics and other sciences, which treat about movement of any bodies" — see [3]. Unfortunately, this point of view dominated in science down to 1925, when it finally had failed, and Newtonian mechanics was deprived with the status of fundamental science. Certainly, rather continuous period had proceeded to this end, when Newtonian mechanics was not able to describe the important physical concepts. Probably, for the first time this problem has arisen in the investigations of J. Maxwell, when he tried to describe true, i.e. not induced, magnetism, but it was not possible. Finally this period was finished by the creation of quantum mechanics.

Between that, in 1771, L. Euler not only clearly had realized an incompleteness of Newtonian mechanics, but also had indicated the path of its extension. In Newtonian mechanics there is only one form of motion, namely translation motion described by transposition of a body-point in space. However in many natural processes spinor motions play the main role. In such a motion the body-point does not change the position in space, but has own rotation. The spinor motions are the main method of accumulation and preservation of energy in the Nature. Not surprised therefore, that Newtonian mechanics has appeared powerless at the level of the microcosmos, where the spinor motions in essence cannot be ignored. In 1776 Euler publishes memoir "New method of determination of motion of rigid bodies" [4], where two independent Laws of Dynamics are stated for the first time: the equation of balance of momentum and equation of balance of kinetic moment (or moment of momentum in accepted, but unsuccessful, terms). This work opens new era in mechanics. Under an appropriate development of ideas of this work modern physics would look completely differently. Unfortunately, the comprehension of ideas of Euler has taken place only at the last quarter of XX century. At the end of XVIII century only J. Lagrange had realized significance of Euler's work, but he had not agreed with its main conclusions. In essence problem was reduced to a possibility or impossibility of the proof of the Archimedes law of the lever. Lagrange, as opposed to Euler, considered that the law of the lever is a corollary of the Newton laws. A large part of extensive introduction to the treatise "Analytical mechanics" [5] Lagrange devotes to the proof of the law of the lever. The Lagrange proof looks rather convincingly, but contains an error, which was not trivial for that time. Namely, Lagrange as the principle of the sufficient basis used reasons of a symmetry, which, as it is well known now, are quite capable to replace by itself conservation laws. In particular, the symmetry concerning a turn round some axes is equivalent to the absence of the moment round the same axes. The Lagrange method of description of mechanics has made a great impression on scientific community. The stable, but faulty, point of view had established that Lagrange's mechanics is quite able to replace by itself Newtonian mechanics. Actually mechanics of Lagrange is a rather poor subclass of Newtonian mechanics and it can not be considered as self-sufficient science about natural phenomena. It follows, for example, from the fact that the fundamental concepts like space, time, forces, moments, energy and etc., are not discussed and can not be introduced into consideration in Lagrange's mechanics, where all these concepts are used, but are not determined. Unfortunately, many theorists with a mathematical kind of thinking obviously underestimate importance and complexity of originating here problems. Besides mechanics of Lagrange is not suitable for the description of open systems, what all-real systems are. All said is quite valid with respect to mechanics of Hamilton that has mathematical dignities, but is very poor from a physical point of view. Main defect of Lagrange-Hamilton mechanics is the false impression, created by these theories, about a completeness of classical mechanics in the fundamental plan and, therefore, about its boundedness. Just this false impression has allowed to M. Plank to say the following words [6]: "Today we must recognize that... frameworks of classical dynamics ... have appeared too narrow to envelop all those physical phenomena that do not lend to direct observation by our rough organs of sense... The proof of this conclusion is given to us by the crying contradiction, that come to light in the universal laws of heat radiation, between the classical theory and experience". This point of view had become conventional in physics. The mechanics had evaded from a discussion of these hard questions and continued researches on the important applied problems.

Let's remind one more statement of M. Plank[7]: "The mechanical phenomena, or movements of material points, and all set of the electrical and magnetic phenomena as a single unit are completely separated. This by two area settle (exhausted) all physics, as all other parts of physics — acoustics, optics, and heat — can be quite reduced on the mechanics and electrodynamics. Final association of these two last classes of the phenomena, that would present by itself the crown of a building of theoretical physics, still it is necessary to give to the future". This statement by M. Plank causes some objections. First of all, the mechanical phenomena are not reduced at all to movements of material points, i.e. to Newtonian mechanics. Leonard Euler proved the basic incompleteness of Newtonian mechanics in 1776. Further, satisfactory theory of the electromagnetic phenomena is not developed till now. At last, more than doubtfully, that association of these theories (even if they would exist) would be a completion of theoretical physics, for obviously there are phenomena leaving the frameworks of these theories in their modern kind. Nevertheless, problem of association of the mechanics and electrodynamics, specified by M. Plank, exists and should be solved. The situation existing in a mechanics and physics can be called paradoxical. On the one hand, there are actual phenomena, which can not be circumscribed within the framework of classical mechanics from the point of view of the first principles. On the other hand, nobody has shown an inaccuracy of these principles. From this it follows, that the principles of Newtonian mechanics are necessary, but not sufficient, for the full description of the known experimental facts. This means, that Newtonian mechanics should be extended by adding of new principles. The statement of these new principles should emanate from intuitive understanding of a nature of those phenomena, which can not be circumscribed by methods of Newtonian mechanics. Certainly, this very complex problem can not be solved by simple means and requires special researches. If the mechanics does not realize necessity of the indicated researches and will limit by the analysis traditional (let even very important) problems, then its future has not any perspectives. If someone doubts of this, he should pay attention to the prompt vanishing of mechanics in the educational and research programs at the end of XX century.

The present article grows out desires of the author to understand the electrical and magnetic phenomena from the point of view of the principles of mechanics. The analysis of the known facts has shown, that the spinor motions,which are absent in Newtonian mechanics, are necessary for a description of the electromagnetic phenomena. Because of this the author thought that the full description of electromagnetism could be executed in the frameworks of Eulerian mechanics. The brief exposition of the main principles of Eulerian mechanics can be found in the paper [8], where the spinor motion is entered in terms of a tensor of turn. The main properties and various representations of the tensor of turn are stated in [9, 10]. The description of classical electrodynamics is given in terms of mechanics in the paper [11]. Besides in [11] the explanation of the known fact about inapplicability of classical electrodynamics for the description of a structure of atom is given. Consequent investigations have shown necessity of an introduction of multi-spin particles. In other words, Eulerian mechanics requires some additions to describe a presupposed structure of atom. Said above, probably, explains a title of the given work. Nevertheless, when describing the main results, the author considers only the pure mechanical aspects, since electromagnetic interpretations of these results are far from a desirable definiteness up to now.

In order to avoid a misunderstanding let us consider the basic terms. In what follows all considerations take place with respect to an inertial system of reference[13].

Newton's Mechanics contains the laws of dynamics of spinless particles. The state of the particle is defined by the vector of position $\mathbf{R}(t)$, the vector of momentum $m\mathbf{R}(t)$, the total energy $U = K + const$, where $K = 0.5m\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}$ is the kinetic energy. The change of the momentum is determined by the vector of force $\mathbf{F}$. Besides, there are derived quantities: the vector $\mathbf{R}\times m\dot{\mathbf{R}}$ is called the moment of momentum of the particle about the origin, the vector $\mathbf{R}\times\mathbf{F}$ is called the moment of the force $\mathbf{F}$. In Newton's Mechanics only so called central forces are admissible. The basic model of Newton's Mechanics is the harmonic oscillator. The basic equation of the simplest form is

$$m\ddot{\mathbf{R}}+c\mathbf{R} = \mathbf{0}. \tag{1}$$

There is no need to speak about other aspects of Newton's Mechanics.

Euler's Mechanics contains the laws of dynamics of single-spin particles. The motion of the single-spin particle is defined by the vector of position $\mathbf{R}(t)$ and by the tensor of turn $\mathbf{P}(t)$. The velocities can be found from the equations

$$\mathbf{V}(t) = \dot{\mathbf{R}}(t), \qquad \dot{\mathbf{P}}(t) = \boldsymbol{\omega}(t)\times\mathbf{P}(t), \tag{2}$$

where the second equation is called the Poisson equation [8]. The total energy U of the particle is the sum $U = K + const$, where the kinetic energy is determined by the quadratic form

$$K = \frac{1}{2}m\mathbf{V}\cdot\mathbf{V} + \mathbf{V}\cdot\mathbf{P}\cdot\mathbf{B}\cdot\mathbf{P}^T\cdot\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^T\cdot\boldsymbol{\omega}, \tag{3}$$

where tensors of second rank $\mathbf{B}$ and $\mathbf{C} = \mathbf{C}^T$ are the inertia tensors in the reference position, the scalar m is the mass. Now we are able to introduce the vector of momentum $\mathbf{K}_1$ and the vector of kinetic moment $\mathbf{K}_2$ by means of the expressions

$$\mathbf{K}_1 = \partial K/\partial\mathbf{V} = m\mathbf{V} + \mathbf{P}\cdot\mathbf{B}\cdot\mathbf{P}^T\cdot\boldsymbol{\omega}, \tag{4}$$

$$\mathbf{K}_2 = \underline{\mathbf{R}\times\mathbf{K}_1} + \partial K/\partial\,\boldsymbol{\omega} = \underline{\mathbf{R}\times\mathbf{K}_1} + \mathbf{V}\cdot\mathbf{P}\cdot\mathbf{B}\cdot\mathbf{P}^T + \mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^T\cdot\boldsymbol{\omega}, \tag{5}$$

where the underlined term is called the moment of momentum. In Euler's Mechanics the change of momentum is determined by the force $\mathbf{F}$ and the change of kinetic moment is determined by the vector of moment $\mathbf{M}$

$$\mathbf{M} = \mathbf{R} \times \mathbf{F} + \mathbf{L}, \tag{6}$$

where the vector $\mathbf{L}$ is called the torque. In general case the torque can't be determined in terms of $\mathbf{F}$. The first and the second laws of dynamics [4] in Euler's Mechanics have the form

$$\dot{\mathbf{K}}_1 = \mathbf{F}, \qquad \dot{\mathbf{K}}_2 = \mathbf{M} = \mathbf{R} \times \mathbf{F} + \mathbf{L}. \tag{7}$$

The more detailed definitions may be found in the paper[8]. The basic model in Euler's Mechanics is the model of rigid body oscillator. In the simplest case the equations of motion of rigid body oscillator, i.e. the rigid body on an elastic foundation, can be derived from the equations (4) − (7) under some assumptions about the elastic foundation [14]. These equations have the form [14]

$$A\,\dot{\boldsymbol{\omega}} + c\boldsymbol{\theta} = 0, \quad \dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \frac{1}{2}\,\boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1-g}{\theta^2}\,\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega}), \quad g = \frac{\theta \sin\theta}{2\,(1-\cos\theta)}, \tag{8}$$

where $\boldsymbol{\theta}$ is a vector of turn [9, 10, 14] $\theta = |\boldsymbol{\theta}|$. We see that even in the simplest case equation (8) has much more complex form then equation (1) for a usual oscillator. However for the plane vibrations we have $\boldsymbol{\theta} \times \boldsymbol{\omega} = 0$. In such a case equation (8) coincides with equation (1). Note that equation (8) corresponds to rotational degrees of freedom only, i.e. the body has a fixed point. In general case we have some combination of equations like (1) and (8).

When speaking about Euler's Mechanics it is necessary to point out the contribution of C. Truesdell [15, 16] who had studied Euler's works published after 1766 and had made them the property of scientific community.

Mechanics of multi-spin particles will be considered in the next sections of the paper.

2 Kinematics and Dynamical Structures of the Multi-Spin Particle

The multi-spin particle A is the complex object consisting of a carrier body A_1 and rotors $A_i \quad (i = 2, 3, ..., N)$ inside of A_1. Let $\mathbf{R}_i \quad (i = 1, 2, ..., N)$ be the position vectors of the mass center of A_i and m_i is the mass of the particle A_i. Let's accept that the set of points $\mathbf{R}_i$ is a rigid body. Let $\mathbf{P}_i$ be the turn-tensors of the bodies A_i. Then we have

$$\mathbf{R}_i = \mathbf{R} + \mathbf{P}_1 \cdot \boldsymbol{\rho}_i, \quad \mathbf{R} = \frac{1}{m}\sum_{i=1}^{N} m_i \mathbf{R}_i, \quad m = \sum_{i=1}^{N} m_i, \tag{9}$$

where $\mathbf{R}$ is the center of mass of A, the vectors $\boldsymbol{\rho}_i$ are the position vectors of the mass center of A_i with respect to the point $\mathbf{R}$ in the reference position. Thus the motion of a multi-spin particle A is determined in terms of $3(N+1)$ scalar functions

$$\mathbf{R}(t), \quad \mathbf{P}_1(t), \quad \mathbf{P}_2(t), \quad ... \quad \mathbf{P}_N(t). \tag{10}$$

The velocities of the multi-spin particle are determined from the next equations

$$\mathbf{V}(t) = \dot{\mathbf{R}}(t), \quad \dot{\mathbf{P}}_i(t) = \omega_i(t) \times \dot{\mathbf{P}}_i(t). \tag{11}$$

We shall consider that rotor A_i is the body of rotation with the axis of symmetry $\mathbf{n}'_i$, which is supposed to be fixed with respect to the carrier body A_1. Because of this we have

$$\mathbf{n}'_i = \mathbf{P}_1 \cdot \mathbf{n}_i, \quad i = 2, ..., N, \tag{12}$$

where $\mathbf{n}_i$ are determined in the reference position. The turn-tensor of the carrier body can be represented in many different, but equivalent, forms

$$\begin{aligned} &\mathbf{P}_1 = \mathbf{T}_2 \cdot \mathbf{Q}(\psi_2 \mathbf{n}_2) = \mathbf{T}_3 \cdot \mathbf{Q}(\psi_3 \mathbf{n}_3) = ... = \mathbf{T}_N \cdot \mathbf{Q}(\psi_N \mathbf{n}_N) \Rightarrow \\ &\mathbf{T}_i = \mathbf{P}_1 \cdot \mathbf{Q}^T(\psi_i \mathbf{n}_i), \end{aligned} \tag{13}$$

where $\mathbf{Q}(\psi_i \mathbf{n}_i)$ is the turn around axis $\mathbf{n}_i$ by the angle ψ_i, $\mathbf{T}_i$ is the turn around axis orthogonal to the axis $\mathbf{n}_i$. For the turn-tensors $\mathbf{P}_i$ we have the analogous expressions

$$\mathbf{P}_i = \mathbf{S}_i \cdot \mathbf{Q}(\varphi_i \mathbf{n}_i). \tag{14}$$

Since the axes $\mathbf{n}_i$ are fixed with respect to the carrier body A_1 we have the conditions

$$\mathbf{T}_i = \mathbf{S}_i \Rightarrow \mathbf{S}_i = \mathbf{P}_1 \cdot \mathbf{Q}^T(\psi_i \mathbf{n}_i).$$

Now the equations (14) take a form

$$\begin{aligned} &\mathbf{P}_i = \mathbf{P}_1 \cdot \mathbf{Q}^T(\psi_i \mathbf{n}_i) \cdot \mathbf{Q}(\varphi_i \mathbf{n}_i) = \mathbf{P}_1 \cdot \mathbf{Q}(\beta_i \mathbf{n}_i), \\ &\beta_i = \varphi_i - \psi_i, \quad i = 2, 3, ..., N, \end{aligned} \tag{15}$$

where β_i is the angle of the turn of the rotor A_1 with respect to the carrier body A_1. Thus we see that the motion of the multi-spin particle can be described in terms of $6 + N - 1$ scalar function, i.e. it has $N + 5$ degrees of freedom. In what follows we shall accept

$$\mathbf{P} \triangleq \mathbf{P}_1, \quad \boldsymbol{\omega} \triangleq \boldsymbol{\omega}_1. \tag{16}$$

Making use of (15) one can find

$$\boldsymbol{\omega}_i = \boldsymbol{\omega} + \mathbf{P}\cdot\dot{\beta}_i \mathbf{n}_i = \boldsymbol{\omega} + \dot{\beta}_i \mathbf{n}'_i, \quad \mathbf{n}'_i = \mathbf{P} \cdot \mathbf{n}_i, \quad i = 2, 3, ..., N. \tag{17}$$

Let's define the kinetic energy K_i of the body A_i by the quadratic form

$$K_i = \frac{1}{2} m_i \dot{\mathbf{R}}_i \cdot \dot{\mathbf{R}}_i + \frac{1}{2} \boldsymbol{\omega}_i \cdot \mathbf{P}_i \cdot \mathbf{C}_i \cdot \mathbf{P}_i^T \cdot \boldsymbol{\omega}_i, \tag{18}$$

where $\mathbf{C}_i$ is the central tensor of inertia of the body A_i in the reference positions. For the rotors we have

$$\mathbf{C}_i = \lambda_i \mathbf{n}_i \otimes \mathbf{n}_i + \mu_i (\mathbf{E} - \mathbf{n}_i \otimes \mathbf{n}_i), \quad i = 2, 3, ..., N, \tag{19}$$

where λ_i, μ_i are the axial central moment of inertia and the equatorial central moment of inertia of the rotor A_i respectively. From the equations (15) and (19) it follows

$$\mathbf{P}_i \cdot \mathbf{C}_i \cdot \mathbf{P}_i^T = \mathbf{P} \cdot \mathbf{C}_i \cdot \mathbf{P}^T. \tag{20}$$

From the equation (9) it follows

$$\mathbf{V}_i = \dot{\mathbf{R}_i} = \mathbf{V} + \omega \times (\mathbf{R}_i - \mathbf{R}) . \tag{21}$$

The momentum $\mathbf{K}_{1i}$ of the body A_i is defined as

$$\mathbf{K}_{1i} = \frac{\partial K_i}{\partial \mathbf{V}_i} = m_i \mathbf{V}_i = m_i (\mathbf{V} + \boldsymbol{\omega} \times (\mathbf{R}_i - \mathbf{R})) = m_i \mathbf{V} + \mathbf{B}_i \cdot \boldsymbol{\omega},$$
$$\mathbf{B}_i = m_i (\mathbf{R} - \mathbf{R}_i) \times \mathbf{E}. \tag{22}$$

The momentum $\mathbf{K}_1$ of the multi-spin particle is defined by the expression

$$\mathbf{K}_1 = \sum_{i=1}^{N} \mathbf{K}_{1i} = m\mathbf{V} + \left(\sum_{i=1}^{N} \mathbf{B}_i \right) \cdot \boldsymbol{\omega} = m\mathbf{V}, \quad \sum_{i=1}^{N} \mathbf{B}_i = \mathbf{0}. \tag{23}$$

The second equality in (23) follows from (9). Let's calculate the kinetic moment $\mathbf{K}_{2i}$ of A_i

$$\mathbf{K}_{2i} = \mathbf{R}_i \times \mathbf{K}_{1i} + \frac{\partial K_i}{\partial \boldsymbol{\omega}_i}, \tag{24}$$

where the first term in the right side is called the moment of momentum and the second term will be called the dynamical spin or the own moment of momentum of A_i. Making use of the formulae (22), (18), (9), (17), (19), (20) one can obtain

$$\mathbf{K}_{2i} = m_i \mathbf{R}_i \times \mathbf{V} + \left(\mathbf{R}_i \times \mathbf{B}_i + \mathbf{P} \cdot \mathbf{C}_i \cdot \mathbf{P}^T \right) \cdot \boldsymbol{\omega} + \lambda_i \dot{\beta}_i \mathbf{n}_i', \quad \beta_1 = 0. \tag{25}$$

The kinetic moment of the multi-spin particle $\mathbf{K}_2$ is defined by the expression

$$\mathbf{K}_2 = \sum_{i=1}^{N} \mathbf{K}_{2i} = \mathbf{R} \times m\mathbf{V} + \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^T \cdot \boldsymbol{\omega} + \sum_{i=2}^{N} \lambda_i \dot{\beta}_i \mathbf{n}_i', \tag{26}$$

where the tensor $\mathbf{C}$ has a form

$$\mathbf{C} = \sum_{i=1}^{N} \left(\mathbf{C}_i - m_i (\mathbf{r} - \mathbf{r}_i) \times \mathbf{E} \times (\mathbf{r} - \mathbf{r}_i) \right), \tag{27}$$

and the vectors $\mathbf{r}$ and $\mathbf{r}_i$ determine the mass centers of the particle A and of the bodies A_i in the reference position respectively. The total kinetic energy of the multi-spin particle is determined by the expression

$$K = \frac{1}{2} m\mathbf{V} \cdot \mathbf{V} + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^T \cdot \boldsymbol{\omega} + \frac{1}{2} \sum_{i=2}^{N} \lambda_i \left(\dot{\beta}_i^2 + 2\dot{\beta}_i \boldsymbol{\omega} \cdot \mathbf{n}_i' \right). \tag{28}$$

Now we are able to write down the laws of motion of the multi-spin particle.

3 The Laws of Motion of a Multi-Spin Particle

The multi-spin particle has $N+5$ degrees of freedom. Thus we need to formulate $N+5$ equations to find the next unknown functions

$$\mathbf{R}(t), \quad \mathbf{P}(t), \quad \beta_i(t), \quad i = 2, 3, ..., N. \tag{29}$$

First of all, we must formulate the laws of dynamics by Euler.

The equation of the momentum balance or the first law of dynamics by Euler

$$\dot{\mathbf{K}}_1 = \mathbf{F}, \tag{30}$$

where $\mathbf{K}_1$ is determined by the expression (23), and the vector $\mathbf{F}$ is the force acting on the multi-spin particle.

The equation of the kinetic moment balance or the second law of dynamics by Euler

$$\dot{\mathbf{K}}_2 = \mathbf{R} \times \mathbf{F} + \mathbf{L}, \tag{31}$$

where the vector $\mathbf{L}$ is called the torque and in general case it can't be defined in terms of a force. The equations (30) and (31) give to us 6 equations. Thus we need to formulate $N-1$ additional equations. For this end let us consider

The equations of motions of the rotors A_i

$$\dot{\mathbf{K}}_{1i} = \mathbf{F}_i, \quad \dot{\mathbf{K}}_{2i} = \mathbf{R}_i \times \mathbf{F}_i + \mathbf{L}_i, \quad i = 2, 3, ..., N \tag{32}$$

where $\mathbf{F}_i$ and $\mathbf{L}_i$ are the force and the torque acting on the rotor A_i from the carrier body A_1. Let's represent the torque $\mathbf{L}_i$ in the next form

$$\mathbf{L}_i = L_{mi}\mathbf{n}'_i + \mathbf{L}^*_i, \quad \mathbf{n}'_i \cdot \mathbf{L}^*_i = 0, \quad L_{mi} = -\eta_i\left(\dot{\beta}_i - \omega_i\right), \quad \eta_i > 0, \tag{33}$$

where $\omega_i = \mathrm{const}$ and $\eta_i = \mathrm{const}$ are the parameters of the particle. Making use of the results of the previous section one can obtain

$$\dot{\mathbf{K}}_{2i} = \mathbf{R}_i \times \mathbf{F}_i + \left((\lambda_i - \mu_i)(\boldsymbol{\omega} \cdot \mathbf{n}'_i)\,\mathbf{n}'_i + \mu_i\boldsymbol{\omega} + \lambda_i\dot{\beta}_i\mathbf{n}'_i\right)^{\cdot}. \tag{34}$$

Substituting this expression into the second equation (32) and multiplying the resulting equation by the vector $\mathbf{n}'_i$ we obtain the additional $N-1$ equations

$$\lambda_i\left(\dot{\beta}_i + \boldsymbol{\omega} \cdot \mathbf{n}'_i\right)^{\cdot} + \eta_i\left(\dot{\beta}_i - \omega_i\right) = 0, \quad i = 2, 3, ..., N. \tag{35}$$

The equations (30), (31), (35) give to us the complete system of equations of motion for the multi-spin particle.

4 The Equation of the Energy Balance

Let us formulate the third fundamental law, i.e. the equation of the energy balance

$$(K + U_p)^{\cdot} = \mathbf{F} \cdot \mathbf{V} + \mathbf{L} \cdot \boldsymbol{\omega} + \delta, \tag{36}$$

where U_p is an intrinsic energy of the particle, δ is the velocity of the energy input into the particle. In what follows we shall consider that $U_p = \mathrm{const}$. This means that the particle does not contain the elastic elements. In the considered case it is easy to calculate δ. Indeed, multiplying (30) by the vector V and so on we obtain

$$\dot{K} = \mathbf{F} \cdot \mathbf{V} + \mathbf{L} \cdot \boldsymbol{\omega} - \sum_{i=2}^{N} \eta_i \dot{\beta}_i \left(\dot{\beta}_i - \omega_i \right). \tag{37}$$

From the comparison of the equations (37) and (36) we see

$$\delta = - \sum_{i=2}^{N} \eta_i \dot{\beta}_i \left(\dot{\beta}_i - \omega_i \right). \tag{38}$$

The quantity δ is generated by the external supply of energy, for example, by the electrical device.

5 Continuum of the Multi-Spin Particles. The law of the Particle Conservation

Let's consider some inertial system of reference. Let Z be a set of the multi-spin particle. Let V be some domain that is fixed with respect to the system of reference. The boundary of V is the closed surface $S = \partial V$. Let $\rho(\mathbf{x},t)$ be a number of the particles in the infinitely small neighborhood of the point $\mathbf{x} \subset V$ at the actual instant of time t

$$\rho(\mathbf{x},t) \geq 0. \tag{39}$$

Let's formulate the law of the conservation of the particles

$$\frac{d}{dt} \int\limits_{(V)} \rho(\mathbf{x},t)\, dV = - \int\limits_{(S)} \rho \mathbf{n} \cdot \mathbf{V} dS, \qquad \int\limits_{(S)} \mathbf{n} \cdot (\rho \mathbf{V})\, dS = \int\limits_{(V)} \nabla \cdot (\rho \mathbf{V})\, dV. \tag{40}$$

From this it follows

$$\frac{d\rho}{dt} + \nabla \cdot (\rho \mathbf{V}) = 0. \tag{41}$$

This is a local form of the law of the particles conservation.

6 The First Law of Dynamics by Euler

The domain V contain the next quantity of the momentum

$$\mathbf{K}_1^* = \int\limits_{(V)} \rho(\mathbf{x},t)\, \mathbf{K}_1(\mathbf{x},t)\, dV(\mathbf{x}), \tag{42}$$

where the momentum of a particle $\mathbf{K}_1$ is defined by the expression (23). Then the first law of dynamics can be written in the form

$$\frac{d}{dt}\int\limits_{(V)} \rho \mathbf{K}_1 \, dV = \int\limits_{(V)} \rho \mathbf{F} \, dV + \int\limits_{(S)} \mathbf{T}_{(n)} \, dS - \int\limits_{(S)} \rho \, (\mathbf{n} \cdot \mathbf{V}) \, \mathbf{K}_1 \, dV.$$

For the last term we have

$$\int\limits_{(S)} \mathbf{n} \cdot (\rho \mathbf{V} \otimes \mathbf{K}_1) \, dS = \int\limits_{(V)} \nabla \cdot (\rho \mathbf{V} \otimes \mathbf{K}_1) \, dV. \tag{43}$$

Now the first law can be rewritten in such a form

$$\underbrace{\frac{d}{dt}\int\limits_{(V)} \rho \mathbf{K}_1 \, dV}_{O(\varepsilon^3)} = \underbrace{\frac{d}{dt}\int\limits_{(V)} [\rho \mathbf{F} - \nabla \cdot (\rho \mathbf{V} \otimes \mathbf{K}_1)] \, dV}_{O(\varepsilon^3)} + \underbrace{\int\limits_{(S)} \mathbf{T}_{(n)} \, dS}_{O(\varepsilon^2)}.$$

From this we see that the next equation is valid

$$\int\limits_{(S)} \mathbf{T}_{(n)} \, dS = \mathbf{0}. \tag{44}$$

Making use of standard considerations the stress tensor can be introduced

$$\mathbf{T}_{(n)} = \mathbf{n} \cdot \mathbf{T}. \tag{45}$$

Thus we have

$$\int\limits_{(V)} \left[(\rho \mathbf{K}_1)^{\cdot} - \rho \mathbf{F} + \nabla \cdot (\rho \mathbf{V}) \, \mathbf{K}_1 + \rho \mathbf{V} \cdot \nabla \mathbf{K}_1 - \nabla \cdot \mathbf{T} \right] dV = \mathbf{0}.$$

In the local form the first law can be represented as

$$\nabla \cdot \mathbf{T} + \rho \mathbf{F} = \rho(\dot{\mathbf{K}}_1 + \mathbf{V} \cdot \nabla \mathbf{K}_1), \qquad \mathbf{K}_1 = m \mathbf{V}(\mathbf{x}, t), \tag{46}$$

where $m = const$ is the mass of the particle that is placed in the point $\mathbf{x}$ at the actual instant of time. The quantity ρm is the mass density. In the right-hand side of the first equation (46) the material derivative of $\mathbf{K}_1$ is written.

7 The Second Law of Dynamics by Euler

The equation of the balance of the kinetic moment in the integral form can be written as

$$\frac{d}{dt}\int\limits_{(V)} \rho \mathbf{K}_2 \, dV = \int\limits_{(V)} \rho \, (\mathbf{R} \times \mathbf{F} + \mathbf{L}) \, dV + \int\limits_{(S)} \left(\mathbf{R} \times \mathbf{T}_n + \mathbf{M}_{(n)} \right) dS - \int\limits_{(S)} \rho \, (\mathbf{n} \cdot \mathbf{V}) \, \mathbf{K}_2 \, dS. \tag{47}$$

where $\mathbf{K}_2$ is defined by the expression (26), $\mathbf{L}$ is a mass density of the external torque. By means of the standard consideration it is easy to derive the Cauchy formula

$$\mathbf{M}_{(n)} = \mathbf{n} \cdot \mathbf{M}. \tag{48}$$

and the local form of the second law

$$\nabla \cdot \mathbf{M} + \mathbf{T}_{\times} + \rho \mathbf{L} = \rho(\dot{\mathbf{K}}_2 + \mathbf{V} \cdot \nabla \mathbf{K}_2), \tag{49}$$

where $\mathbf{K}_2$ is the dynamical spin of a particle

$$\mathbf{K}_2 = \mathbf{P}(\mathbf{x},t) \cdot \mathbf{C} \cdot \mathbf{P}^T(\mathbf{x},t) \cdot \boldsymbol{\omega}(\mathbf{x},t) + \sum_{i=2}^{N} \lambda_i \dot{\beta}_i(\mathbf{x},t)\, \mathbf{n}_i'(\mathbf{x},t), \tag{50}$$

and the tensor $\mathbf{C}$ is defined by the expression (27). To this equation it is necessary to add the equation of motion for the rotors (35)

$$\lambda_i \left(\dot{\beta}_i(\mathbf{x},t) + \omega(\mathbf{x},t) \cdot \mathbf{n}_i'(\mathbf{x},t) \right)^{\cdot} + \eta_i \left(\dot{\beta}_i(\mathbf{x},t) - \omega_i(\mathbf{x}) \right) = 0, \quad i = 2, 3, ..., N. \tag{51}$$

8 The Equation of the Energy Balance

Let's introduce the total energy in the domain V

$$E = \int_{(V)} \rho(\mathcal{K} + \mathcal{U}) dV, \tag{52}$$

where $\mathcal{K}, \mathcal{U}$ are the density of the kinetic and intrinsic energy respectively.

The equation of the energy balance is the next statement

$$\frac{d}{dt} \int_{(V)} \rho(\mathcal{K} + \mathcal{U}) dV = \int_{(V)} \rho \left[\mathbf{F} \cdot \mathbf{V} + \mathbf{L} \cdot \boldsymbol{\omega} + q \right] dV +$$
$$\int_{(S)} \left(\mathbf{T}_n \cdot \mathbf{V} + \mathbf{M}_n \cdot \boldsymbol{\omega} + h_n \right) dS - \int_{(S)} \rho \mathbf{n} \cdot \mathbf{V} (\mathcal{K} + \mathcal{U}) dS, \tag{53}$$

where

$$h_n = \mathbf{n} \cdot \mathbf{h}. \tag{54}$$

The equation of the energy balance in local form can be written as

$$\rho \left[\frac{d\mathcal{U}}{dt} + \mathbf{V} \cdot \nabla \mathcal{U} \right] = \mathbf{T}^T \cdot\cdot (\nabla \mathbf{V} + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{M}^T \cdot\cdot \nabla \boldsymbol{\omega} + \boldsymbol{\nabla} \cdot \mathbf{h} + \rho q. \tag{55}$$

9 Continuum by Lord Kelvin

Let's accept the next assumptions

$$\mathbf{V} = \mathbf{0}, \quad \mathbf{T} = \mathbf{0} \quad \Rightarrow \quad \rho = \mathrm{const}. \tag{56}$$

In such a case the second law by Euler takes a form

$$\nabla \cdot \mathbf{M} + \rho \mathbf{L} = \rho \dot{\mathbf{K}}_2. \tag{57}$$

The equation of the energy balance can be simplified as well

$$\rho \frac{d\mathcal{U}}{dt} = \mathbf{M}^T \cdot\cdot \nabla \boldsymbol{\omega} + \nabla \cdot \mathbf{h} + \rho q. \tag{58}$$

Let's accept one more restriction

$$\mathbf{M} = \mathbf{H} \times \mathbf{I} \quad \Rightarrow \quad \nabla \cdot \mathbf{M} = \nabla \times \mathbf{H}, \tag{59}$$

where $\mathbf{I}$ is unit tensor. Then equation of the energy balance (58) for the isotermic processes takes a form

$$\rho \frac{d\mathcal{U}}{dt} = -\mathbf{H} \cdot \boldsymbol{\nabla} \times \boldsymbol{\omega}. \tag{60}$$

For the kinetic moment we accept the notation

$$c^{-1}\mathbf{E} = \rho \mathbf{K}_2 = \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^T \cdot \boldsymbol{\omega} + \lambda_0 \dot{\beta}_0 \mathbf{P} \cdot \mathbf{n}, \quad c = \mathrm{const}, \tag{61}$$

where

$$\mathbf{C} = \lambda \mathbf{n} \otimes \mathbf{n} + \mu(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}). \tag{62}$$

Making use of (59) and (61) equation (57) can be replaced by

$$\boldsymbol{\nabla} \times \mathbf{H} + \rho \mathbf{L} = \frac{1}{c} \frac{d}{dt} \mathbf{E}. \tag{63}$$

This equation has a form of the first Maxwell equation. Let the turn-tensor $\mathbf{P}$ be represented in the form

$$\mathbf{P} = \mathbf{Q}(\boldsymbol{\theta}) \cdot \mathbf{Q}(\varphi \mathbf{n}), \quad \boldsymbol{\theta} \cdot \mathbf{n} = 0, \tag{64}$$

where the vector $\boldsymbol{\theta}$ and the angle of own rotation are supposed to be small. In such a case we have

$$\boldsymbol{\omega} = \dot{\boldsymbol{\theta}} + \dot{\varphi} \mathbf{n} = \dot{\boldsymbol{\vartheta}}, \quad \boldsymbol{\vartheta} = \boldsymbol{\theta} + \varphi\, \mathbf{n}. \tag{65}$$

Equation (60) can be rewritten now as

$$\rho \frac{d\mathcal{U}}{dt} = -\mathbf{H} \cdot \frac{d}{dt} \boldsymbol{\nabla} \times \boldsymbol{\vartheta}. \tag{66}$$

Let's accept the simplest representation for the specific internal energy

$$\rho \mathcal{U} = \frac{1}{2} \kappa |\boldsymbol{\nabla} \times \boldsymbol{\vartheta}|^2, \quad \kappa = \mathrm{const}. \tag{67}$$

Then for the vector $\mathbf{H}$ we obtain

$$\mathbf{H} = -\kappa \boldsymbol{\nabla} \times \boldsymbol{\vartheta}. \tag{68}$$

If we take into account equality (62) then expression (61) can be rewritten as

$$c^{-1}\mathbf{E} = \mu\dot{\boldsymbol{\theta}} + (\lambda\dot{\varphi} + \lambda_0\dot{\beta})\,\mathbf{n} + \lambda_0\dot{\beta}\boldsymbol{\theta} \times \mathbf{n}. \tag{69}$$

In order to get the simplest case, let's accept the restrictions

$$\lambda = \mu, \quad \lambda_0 = 0. \tag{70}$$

Then we obtain

$$\mathbf{E} = \mu c \frac{d}{dt}\boldsymbol{\vartheta}. \tag{71}$$

From equations (68) and (71) it follows

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{d}{dt}\mathbf{H}. \tag{72}$$

Let's write down equation (63) and (72) together

$$\boldsymbol{\nabla} \times \mathbf{H} + \rho\mathbf{L} = \frac{1}{c}\frac{d}{dt}\mathbf{E}, \quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{d}{dt}\mathbf{H}, \qquad \kappa = \mu c^2. \tag{73}$$

The obtained equations are the classical equations by Maxwell.

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A.I. Lurie — Works on Mechanics*

Abstract

The report is devoted to contribution by A.I. Lurie in the development of mechanics in Russia. It is necessary to underline, that the scientific interests of A.I. Lurie were extremely wide and concerned to different fields of mechanics and the control process. The books and textbooks by A.I. Lurie, which were studied by hundred thousand of students, engineers and scientific workers, show high samples of scientific creativity. In 1927 the Leningrad mechanical society was founded, which has played an important role in the development of mechanics in USSR. The one of organizers of this society was A.I. Lurie. Among many achievements of the society there was organization of the issuing of well known journal "Applied Mathematics and Mechanics". A.I.Lurie was an editor of translations of many remarkable books on a mechanics. A.I. Lurie is recognized by the scientific community as the distinguished scientist - encyclopedist. A.I. Lurie was by a member of National Committee of USSR on theoretical and applied mechanics, and in 1961 he was selected by corresponding member Academy of Sciences of USSR. The name of A.I. Lurie has come in the history of mechanics in Russia for ever.

1 Introduction

A man cannot choose his birthday or birth place. However, time and habitation country significantly influence upon the making of a person and determine the character of his activity. Nevertheless, at all times and in all countries the individuals are born, who are realized as self-independent and self-sufficient creatures. Such persons play the role of "evolution catalysts" for the society, into which they are involved. The problems solved by them are never accidental but determined by the higher necessities of the society. The main feature of a realized individual is a capacity of a person not only to perceive intuitively the society higher necessities but to take them as a guide to the action. Therefore, this is impossible to make a correct evaluation of a contribution of any person into the evolution of a community (either of its certain part) if one does not realize clearly the state of this community and its necessities at the evolution stage considered. No doubt, Anatoly Isakovich Lurie had realized himself as a self-independent individual, whose versatile fruits of work we sense so distinctly. The aim of this presentation is a discussion of

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A.I. Lurie's contribution into evolution of mechanics in Russia. A.I. Lurie had began his self-independent investigations in mechanics in 1925 all at once on graduating from the Faculty of Physics and Mechanics of Leningrad Polytechnical Institute. Think of Russia being in 1925! The previous decade resulted in an extremely hard state for Russia. The First World War, the October Socialist Revolution, and, finally, the fratricidal civil war, which is the worst and the most dangerous among all kinds of wars. All this had led to the scarcity and dissociation of the Russian brain-power to nearly complete destruction of relatively weak industry together with the total absence of finances for purchase of needed equipment. In addition, Russia was, actually, in a complete isolation from the all-world community. Consequently, development of the native industry became one of immediate tasks. Traditionally, only ship building was rather well-developed, but other fields of industry (such as mechanical engineering, power engineering, turbine construction, instrument-making and aircraft industries, etc.), they all were present in embryo. Everything mentioned above had to be built up anew. First of all, tens of thousands of skilled engineers were to be trained. It should be taken into account, that these skilled engineers had to be prepared from a relatively uneducated medium, since schools worked under abnormal conditions in 1914-1922 as well. For training a skilled engineer brain-power competent specialists and, also, text-books were needed. It cannot be said that there were no scientists in the field of mechanics in Russia. Suffice it to recall such first-class scientists as N.E. Zhukovsky, I.G. Bubnov, I.V. Meschersky, A.A. Fridman, A.N. Krylov, P.F. Papkovich, E.L. Nicolai, and many others. However, they were extremely few in number for such a vast country as Russia. As for text-books on mechanics for universities, they were actually absent. Just then, the generation of Russian scientists, to which A.I. Lurie belonged, had to start the work. Creative work of the above-mentioned scientists received a high appraisal by the launching of the first in the world artifical satellite on October 4th, 1957, along with the fact that to 1960 the technical education in Russia was recognized as one of the best in the world by the international community.

On graduating from the Polytechnical Institute A.I. Lurie hold the post of a lecturer at the chair "Theoretical Mechanics" of the institute. Hereafter, A.I. Lurie began his persistent research work. It is necessary to emphasize that A.I. Lurie was utterly interested in various fields of mechanics and of control theory. It is accounted for by the fact that A.I. Lurie was tightly concerned with organizations engaged in development and production of new technique. Among the organizations, Leningradskii Metallicheskii Zavod (a Leningrad Metal Plant), Osoboe Tekhnicheskoe Byuro (the Special Technical Department), and Osoboe Konstruktorskoe Byuro (the Special Constructor Department) to be pointed out in the first place. As it is known, creation of a new technique is accompanied by numerous problems associated with mechanics and the control theory. Over the post-war years, contacts of A.I. Lurie with industrial organizations were essentially widened. Multiform demands of practice made the scientist to perform his investigations simultaneously in various directions. Therefore, in describing the works of A.I. Lurie on mechanics we ought to divide the works into separate groups and to break the chronological succession. As for investigation on the control theory, into which A.I. Lurie made a valuable contribution, they represent the subject of a separate consideration.

2 At the source of the Leningrad School of Mechanics

A.I. Lurie was not only the eminent Scientist, but a striking Teacher as well. He left hundreds of disciples in mechanics, many of them became world-known scientists. Monographs and text-books by A.I. Lurie, by which hundreds of thousands of future engineers studied mechanics, continue to remain brilliant examples of scientific creative work. The scientific style of A.I. Lurie was remarkably rigorous and clear, without any pseudo-scientific excesses. The scientist gave a lot of efforts to the development of a mathematical technique that allowed to solve the problems in the most effective and clear way. In particular, A.I. Lurie was a staunch devotee to the direct tensor calculus, and he made a remarkable contribution into development and introduction of this new technique.

In 1927, Leningrad's Mechanical Society had been established, which played a great role in development of mechanics in the USSR. Prof. E.L. Nicolai was the organizer and the permanent President of the Society, whereas A.I. Lurie was its Scientific Secretary. Among many achievements of the Society, one should point out the organization of edition of the first in the USSR specialized journal on mechanics and applied mathematics. Initially, beginning from 1929, the title of the journal was "Vestnik Mekhaniki i Prikladnoy Matematiki" ("News of Mechanics and Applied Mathematics"). In 1933, the journal was transformed into an all-union periodic edition "Applied mathematics and mechanics" ("Prikladnaya Matematika i Mekhanika"). Up to 1937, when edition of the journal was transferred to Moscow and referred to the Institute of Problems in Mechanics, Russian Academy of Sciences USSR, E.L. Nicolai was appointed as the Editor-in-Chief, whereas A.I. Lurie worked as the Executive Editor.

As it was mentioned above, at the time the special literature on mechanics was, actually, absent in Russia. One had to study mechanics by English, German, and French original publications, which was possible for A.I. Lurie but not for many those, to whom the knowledge of mechanics was necessary. Therefore, there were severe need in edition of scientific literature translations. A.I. Lurie was actively involved in this important work. In particular, a lot of translations of remarkable monographs were edited by the scientist, for instance, such as E. Trefftz, Mathematical theory of elasticity (1934); Hekkeler, Statics of elastic body (1934); P. Pfeiffer, Oscillations of Elastic Bodies (1934); Analytical mechanics by Lagrange (1938); C. Truesdell, First Course in Rational Continuum Mechanics (1975) and so on. Note that in thirties, the translations of foreign editions played a significant role in training the engineers in the USSR.

Scientific merits of A.I. Lurie are universally recognized. The world scientific community knows him as a prominent scientist-encyclopedist. A.I. Lurie was a member of the National Committee of the USSR on Theoretical and Applied Mechanics, and in 1961 Prof. A.I. Lurie was elected as a corresponding member of Academy of Sciences of the USSR. V.V. Novozhilov, Academician of RAS of the USSR, stated that A.I. Lurie was attributed to those selected scholars, for whom the highest scientific titles were their names.

3 Operational calculus

Early works of A.I. Lurie were devoted to hydrodynamics of viscous liquids. This was the subject of his thesis defended in 1929. Generally speaking, no theses were defended at that time. Nevertheless, A.I. Lurie wrote his thesis and it was discussed at the meeting of Academic Council. V.A. Fok and A.A. Satkevich, well-known professors, were the referees. The thesis was defended successfully, and a positive decision was sent to the Council Record Office. In the USSR the advanced degrees were re-established only in 1933, and just at that time A.I. Lurie gained a honorary doctorate. Note that earlier he had attained a Professor title already. Although the works on hydrodynamics of viscous liquids are not among most important achievements of A.I. Lurie, nevertheless, the scholar pioneered in applying an approach based on operational calculus, which was new for this field of mechanics. The approach was approved by Academician V.A. Fok, and he advised A.I. Lurie to continue investigations in this direction. These investigations resulted in publication of the paper "On the theory of the sets of linear differential equations with constant coefficients" (Trudy Leningradskogo Industrialnogo instituta – Transactions of Leningrad Industrial Institute, 1937, No. 6, pp. 31–36) and of a monograph "Operational calculus" (Moscow–Leningrad: ONTI, 1938, in Russian). Later, these investigations were developed and resulted in creation of A.I. Lurie's symbolic method discussed below in section devoted to the elasticity theory.

The idea of operational calculus was proposed by Oliver Heaviside in 1893. Further this idea was extended in works of T.J. Bromwich, E.P. Adams, H. Jeffreys and some other western scientists. Usually, operational methods were applied to the calculation of electric circuits. Meanwhile, at the end of thirties they did not have a wide application in mathematical physics. As H. Jeffreys pointed out, this happened since there were some obscurities in the basic theory and there were no systematic description of operational methods. For the first time, the systematic theory of operational methods was described in the book *Harold Jeffreys. Operational methods in mathematical physics. London, Cambridgde, 1927.* The second edition of the book was published in 1931. We see that in 1930 the operational calculus became rather popular, mainly, in England. Therefore it would not be correct to speak about A.I. Lurie's contribution into operational calculus. The scientist is worthy in another matter. Firstly, the West is the West, whereas Russia of thirties was a country, where it was a great problem to become acquainted with achievements of foreign scientists. Secondly, abstract ideas, let them even be rather perspective, were not completely appropriate for the technical education in Russia of that time. There was a need in convincing applications of the ideas to the concrete technical problems. Just that was made by A.I. Lurie. In particular, two well-known problems were considered in his paper of 1937 mentioned above. The first one was the problem of a body in a flow of viscous liquid. On the basis of operational calculus, there was derived a solution, merely in a few lines, which had been obtained by L.S. Leybenson in 1935 by another method. The second example was a derivation of the general solution for equations of statics in the linear elasticity. This solution, without derivation, was published by B.G. Galerkin in 1930 in Doklady AN SSSR. Earlier B.G. Galerkin made a presentation on this subject at the Meeting of Leningradian Mechanical Society. During the presentation he only wrote the formulae of the solution on the blackboard and suggested to the colleagues to check that the formulae are the solution of equations of statics and allow to satisfy arbitrary boundary conditions. For the first time, A.I. Lurie gave in his paper the complete derivation of Galerkin's solution. In the same paper, he obtains by this method the solution of Lamé dynamic equations, which gives Galerkin's solution as a particular case. In the monograph "Operational calculus" one can find a lot of solved problems which are of independent significance and pronounced technical trend. Owing to the fact, the monograph became a manual for engineers engaged in different organizations related to calculations. The same destiny waited for majority of another works by A.I. Lurie.

In conclusion, we note one more work by A.I. Lurie. Towards the end of the thirties, at Leningradskii Metallicheskii Zavod (Leningrad Metal Plant), in the course on construction of powerful vapor turbines, a phenomenon, new for that time, namely, self-excitation of severe vibrations in high-pressure pipelines, was discovered. Later, the phenomenon was named as the hydrodynamical shock. The vibration occurred to be so active that the walls of a huge workshop stated to shake. A.I. Lurie was involved into solving of problem. He constructed a mathematical model and made the calculations with in collaboration with his coworker A.I. Chekmarev. The observed phenomenon of self-excitation of vibrations in a pipeline was completely borne out by the above calculations. This seemed to be the first solution of a problem of such type. Far later, similar calculations were performed by A.I. Lurie's students and colleagues, namely, V.A. Palmov, A.A. Pervozvansky, V.A. Pupyrev under the guidance of A.I. Lurie for pipelines of another types. For the solution of the problem, the operational calculus was used as well. Difficulty arose with formulation of the criteria of stability. Usual criteria (of Gurvitz, Mikhailov type etc.) were not applicable, since critical values had to be found from the transcendental equation. This problem is not solved yet for the time being for the general case. However, the numerical calculations were completely performed.

4 Analytical mechanics

It was mentioned that just after graduating from the institute A.I. Lurie began to teach theoretical mechanics. At that time a well-known scientist I.V. Mescherskiy hold the Chair of Theoretical Mechanics, who, in addition, was the author of the unique problem book on theoretical mechanics, which, up to date, had gone into 38 editions and it had been translated into many foreign languages. By the way, just I.V. Mescherski pioneered in the introduction of the exercises as a form of education. One should take into account, that at that time in the world there was no such a subject "theoretical mechanics" as an element in the technical education. There existed the subject "analytical mechanics", which was studied at the mathematical faculties of the universities. One can mention famous monographs "Analytical dynamics" by E.T. Whittaker and "Theoretical mechanics" by P. Appel. There were also some other textbooks, but they were not translated into Russian. In 1922 in Russia the lithographic lecture notes on the analytical mechanics by N.V. Roze, a Professor of Leningrad State University, were published. All these monographs were not very appropriate for teaching in technical institutes, from which practical engineers were graduating. Thus there was a vital necessity in such a textbook on theoretical mechanics for technical institutes, a textbook, which could present in an understandable way all the achievements of the theory together with its practical applications. At the time being, when the development of the fundamental mechanics in Russia is on the very high leveln the West, one can hardly imagine all the hugeness of the task confronting the Russian technical education at that time. The slogan "to overtake and surpass!" was not even put on agenda there, and "to overtake!"

slogan seemed to be a far remote dream. A.I. Lurie, being a highly educated person, realized perfectly all this. The scientist not only realized the situation, but also he made every effort to change it. As a result, in 1932–1933 there was published a monograph in three volumes: L.G. Loytsianski, A.I. Lurie "Theoretical mechanics". The first and the second volumes contained the material required for technical education of an engineer, and the third volume contained more sophisticated methods of the analytical mechanics with applications to the large amount of specific problems. Later on, the contents of the first two volumes was accepted as the compulsory program for the technical institutes. All next editions of this text-book did not include the third volume. The sixth (and the last) edition of the monograph had been published in 1983, after the death of A.I. Lurie. This monograph was novel in many relations. Firstly, contrary to its western analogues, it paid a lot of attention to the technical applications. Accordingly, certain theoretical problems of the analytical mechanics, which might "frighten" practical engineers, were omitted. On the contrary, applied aspects were far more extended. Wide applicability of vectorial calculus, nearly unused at that time, gave the additional clarity to the book. A small book by L. Silberstein ("Vectorial mechanics". London: McMillan & Co., 1913) was the only book on mechanics of that time using vectorial calculus. However, the book by Silberstein was absolutely unuseful for the purposes of the technical education, and, apart from that, it used the terminology, which had not been accepted later on. The textbook by L.G. Loytsiansky and A.I. Lurie played a great role in education of Russian engineers. By the way, the theoretical mechanics was one of compulsory courses, not less than of 230 hours, in all technical institutes of Russia. This was one of main reasons of the high educational level of Russian engineers. Regrettably, beginning from the late sixties, the volume of courses on mechanics in many technical institutes has decreased steadily, and this led to the declining of the level of the graduate mechanical engineers. Since the technical education made the foundation for entire higher education in Russia, declining in the professional level of engineers led to the diminution of the IQ of the Russian population as a whole. We compare not more than two numbers characterizing the IQ of the Russian population, namely, in 1960 Russia took the second place among all the world countries with respect to this quality, whereas in 1995 our country was merely at 54th place. Of course, reduction of a role of mechanics in the technical education is not the only reason for such poor situation, but this is one of the main problems. It is out of line here to go to the detailed discussion of such a burning (and not only for Russia) question, however, this is a indisputable fact.

Now, we turn to the description of the creative work of A.I. Lurie. After the publication of the textbook on theoretical mechanics, the scientist's research interests had been concentrated in another fields of mechanics (to be discussed further), for the period of almost 20 years. This statement is not completely true, because during all these years A.I. Lurie gave courses in various areas of mechanics, including the course on analytical mechanics for students of the Faculty of Physics and Mechanics, and, naturally, the scientist continued to cogitate on the analytical mechanics problems. However, his publications of those years were devoted to other problems. At the beginning of the fifties, due to necessities in computation of motion of artificial satellites and solution of some other problems, for instance, in developing the gyroscopic systems, A.I. Lurie interests, again, turned to the analytical mechanics. As a result, in 1961, the fundamental monograph "Analytical Mechanics" was published. It should be noticed that Russia already scored big successes in the field of education at that time. Development of the fundamental mechanics in Russia gained the level of leading countries of the West. As for certain fields in mechanics, for instance, the theory of gyroscopic systems, Russia had taken the leading position. The monograph "Analytical mechanics" to the full extent supports the said above. In the monograph not only all the basic methods of analytical mechanics were presented, but their essential development was made. The monograph, as a whole, held the peculiar features of A.I. Lurie's creative scientific work, such as clarity and laconism in the presentation of a subject along with a high theoretical level and pronounced trend to the applied science. Very few scientists succeeded in performing the synthesis of such a kind. Everyone who encounters a need to solve any problem in the field of dynamics of systems with the finite number degrees of freedom, might be advised to look through, first of all, "Analytical mechanics" of A.I. Lurie. This is very probable that the reader will find the problem needed or something very much alike in this monograph. In many respects the monograph can be named as an encyclopaedia or a reference book. However, in comparison with encyclopaedia or a reference book, all the problems are discussed basing on the same foundation, along with the thorough treatment of all the details there. Of course, this leads to the apparition of new elements in many of these problems. This would take a lot of efforts to describe all these new elements, but no doubt that the reader will discover them easily himself. To illustrate this, we shall mention the description of the relative motion, and make an emphasis on the kinematics of rigid bodies, embedded by rotors (gyroscopes). Below we restrict our description to short references for a few those new elements[1], which are of theoretical significance, i.e. just they make their contribution to the bases of analytical mechanics. Here, first of all, one must mention the description of rotation of a rigid body by means of the vector of finite rotation. The vector, as such, was known long ago. Nevertheless, even in modern textbooks on physics and in some contemporary papers in the field of mechanics the possibility to describe the rotation by means of vector is denied. This fact is caused by the erroneous application of the concept of the superposition of rotations. A detailed mathematical apparatus for effective using of the finite rotation vector is developed in "Analytical mechanics". In particular, the theorem is derived which gives the expression for the finite rotation vector corresponding to the superposition of rotations via finite rotation vectors for the composing rotations. The rule of inversion for the finite rotation vectors is established. The formula giving the relation between the angular velocity vector and the time derivative of the finite rotation vector. The Darboux problem, i.e. the problem for determination of rotations by given vector of angular velocity, is formulated in terms of the finite rotation vector. The formulae giving the relation between the finite rotation vector and the Rodrigo–Hamilton and Cayley–Klein parameters, are established. The significance of the above results is concerned with the fact that they can not become outdated, i.e. once and forever, they have gone into mechanics. One more fundamental result is the following. By the end of the nineteenth century, Rayleigh introduced the concept of the dissipative function as the quadratical form of velocities. This disspative function was very useful in the analysis of nonconservative systems. Regrettably, the Rayleigh dissipative function was defined only for one class of friction, namely, for the linear viscous friction. In the monograph "Analytical mechanics" the concept of the dissipative function was generalized for an arbitrary dependence of friction forces on the velocity. One can find in the

[1] This is hardly possible to propose a lot of such new elements in mechanics

book the examples of the dissipative functions for various friction laws. In particular, the dissipative function for the Coulomb friction law is constructed. Now, the function is widely used in problems of dynamics of the systems with Coulomb friction.

5 The theory of thin elastic shells

A great number of A.I. Lurie's works is devoted to theories of thin rods, plates and shells. In this section, we concentrate our attention on the theory of shells. The theory of shells is one of the most actual directions of research in mechanics. This is caused by the following circumstances. Firstly, thin-walled constructions are widely applied in technology and civil engineering. By the way, the Nature also widely uses thin-walled elements, e.g. biological membranes, in biological systems. Secondly, in the theory of shells the general mechanics, generalized in comparison to the Newtonian mechanics, is developed in the explicit way. A.I. Lurie actively worked on the theory of shells during more than 25 years. His first paper in this field "The investigations on the theory of elastic shells" (Trudy Leningradskogo Industrialnogo Instituta — Transactions of Leningrad Industrial Institute, 1937, No. 6, pp. 37–52) had issued in 1937, and the last one "On the statical geometrical analogy of the theory of shells" was published in 1961. As a whole, A.I. Lurie had published five extensive papers and one monograph. The monograph, "Statics of the thin elastic shells" (Gostekhizdat, 1947, 252 p., in Russian), played an important role of a reliable scientific basis for practical calculations. It was the first monograph by a Russian author specialized in the theory of shells. One should mention that the theory of shells was one of the first[2] areas in mechanics of solids, where, as long ago in 1940, the post-revolutionary Russia not only had achieved the level of well-developed western countries, but even left them behind. The role of A.I. Lurie in this success can not be overemphasized, though, undoubtely, achievements of other Russian scientists, among whom A.L. Goldenweizer and V.V. Novozhilov must be mentioned, are very significant. In the first cited above work A.I. Lurie writes: "As compared to the that, hardly understandable, presentation of the subject given in chapter XXIV of the well-known work by Love[3], we, using the language of the vectorial notation, have simplified essentially all derivations". We emphasize that in this work, as well as in all his works, A.I. Lurie applies the most modern versions of the corresponding mathematical theories. In the above case it was the geometry of surfaces. In the work under discussion A.I. Lurie proposed a rigorous theory for infinitesimal deformations of surfaces for enough general case. At the same time, when deriving the equilibrium equations in displacements, A.I. Lurie applied a bit modified but nevertheless restricted method proposed by B.G. Galerkin two years before. In his work "The general theory of elastic thin shells" (PMM, 1940, IV, N 2, pp. 7–34, in Russian) A.I. Lurie already took off all the restrictions and developed the complete theory based on Kirchoff–Love hypotheses, in terms of tensor calculus. Even at present, the theory by A.I. Lurie can not be improved without abandon the Kirchoff–Love hypotheses. The monograph "Statics of thin elastic shells" is quite characteristic for all the creative work of the scientist. In all his investigations, he never forgot for whom

[2]Among other areas where the priority of Russian scientists is unquestionable, one must point out Kolosov–Muskhelishvili method for a plane elasticity problem.

[3]August Love. Mathematical theory of elasticity. Moscow, Leningrad: ONTI, 1935

his works were written. In the case of the mentioned monograph, he beared in mind numerous groups of engineers engaged in various design and constructor bureau. For this reason the tensor calculus was not used in the book. The presentation is ultimately clear and simple, but also rigorous, and is limited to examination of most usable classes of shells, mainly, by shells with rotational symmetry. In this monograph one can find a great number of solved problems, with easily used in practice design formulae. This is a well-known fact that the equations of the theory of shells are cumbersome, and their solutions are awkward and can be hardly used in engineering calculations. Taking this into account, A.I. Lurie renounced from presentation of exact solutions, which are dubious to the certain degree because of approximate character of the theory of shells itself, and he did apply asymptotical methods. As a result, he succeeded to obtain compact and easy in use formulae for computations. The above mentioned features of the monograph had made it a manual handbook for calculating engineer just after the publication. One would mistaken to believe that the monograph in question is of no other than applied significance. The results presented referred to the most important achievements of the theory. Actually, the theory of shells was inspired by vital practical necessities. Therefore, it would be useless to write down cumbrous equations and even more unmanageable solutions, which appeared often in the beginning of the XX century and were never applied anywhere. The monograph by A.I. Lurie helped the theory of shells to escape this sad destiny. The asymptotical formulae obtained in the monograph were a result of quite rigorous mathematical analysis. One should take into account that the theory of differential equations with an infinitesimal parameter in the coefficients of the highest derivatives had not been developed yet. It had appeared ten years later, and it grew up just from the problems of the theory of shells. Among concrete results discussed in the monograph, one has to point out the problem on the stress concentration in the vicinity of a hole at the surface of the cylindrical shell. The classical Kirsch problem on the stress concentration near the hole in the plane subjected to the extension, is known. The problem solved by A.I. Lurie is a far generalization of the Kirsch problem. Afterwards, this problem gave rise to the separate large part of the theory of shells. Without dwelling on other A.I. Lurie's results in this field, we note that all the six his works on the theory of shells became classical and now they are an integral part of the modern theory of shells.

6 Spatial problems of linear and nonlinear elasticity theories

A.I. Lurie devoted a great number of his scientific works to the spatial problems of the elasticity theory. We concentrate our attention on three of them, namely:

1. Spatial problems of the theory of elasticity. Gostekhizdat, 1955, 491 p.
2. Theory of elasticity. Nauka, 1970, 939 p.
3. Nonlinear theory of elasticity. Nauka, 1980, 512 p.

All the three monographs are related to one and the same area of mechanics. Meanwhile, they are not intercrossed with respect to contents. The first book concerns rigorous solutions for problems on statics of elastic bodies. The attention is mainly focused on analysis of problems for an elastic layer. Just in this monograph A.I. Lurie proposed a new method, which became a widely known as a Lurie symbolic method. This approach is a far extended generalization of operational methods, but it has also essential differences. Let us demonstrate the idea of this method using as an example the problem for an elastic layer. We put down the Lamé static equation for a layer $|z| \leqslant h$:

$$\nabla \cdot \nabla \mathbf{u}(x, y, z) + \frac{1}{1-2\nu} \nabla\nabla \cdot \mathbf{u}(x, y, z) = \mathbf{0}, \qquad |z| < h, \quad x, y \in \Omega, \tag{1}$$

where $\mathbf{u}$ is the displacement vector, and the body forces are omitted for simplicity. The nabla operator can be represented as follows

$$\nabla = \mathbf{k}\,\frac{d}{dz} + \boldsymbol{\Lambda}, \qquad \boldsymbol{\Lambda} = \mathbf{i}\,\frac{d}{dx} + \mathbf{j}\,\frac{d}{dy}, \qquad \mathbf{k} \cdot \boldsymbol{\Lambda} = 0. \tag{2}$$

Substituting equation (2) into equation (1), we rewrite the last one as

$$\frac{d^2}{dz^2}\left(\mathbf{E} + \frac{1}{1-2\nu}\,\mathbf{k}\,\mathbf{k}\right) \cdot \mathbf{u} + \frac{1}{1-2\nu}(\mathbf{k}\,\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\,\mathbf{k}) \cdot \frac{d\mathbf{u}}{dz} + \\ + \boldsymbol{\Lambda} \cdot \boldsymbol{\Lambda}\,\mathbf{u} + \frac{1}{1-2\nu}\,\boldsymbol{\Lambda}\,\boldsymbol{\Lambda} \cdot \mathbf{u} = \mathbf{0}. \tag{3}$$

If we consider operator $\boldsymbol{\Lambda}$ in this equation as a vector, which does not depend on the variable z, then equation (3) is an ordinary differential equation with constant coefficients. Let us add "initial" conditions to equation (3)

$$z = 0: \qquad \mathbf{u} = \mathbf{f}(x, y), \quad \frac{d\mathbf{u}}{dz} = \mathbf{g}(x, y). \tag{4}$$

Thus we obtain the initial value problem (3)–(4), where variables x, y are considered as parameters. Particular solutions of the problem are sought in the form of

$$\mathbf{u} = \exp(i\lambda z)\; \mathbf{a}(x, y). \tag{5}$$

Substituting representation (5) into equation (3) we obtain a homogeneous set of equations for a vector $\mathbf{a}$

$$\left[-\lambda^2\left(\mathbf{E} + \frac{1}{1-2\nu}\,\mathbf{k}\,\mathbf{k}\right) + \frac{i\lambda}{1-2\nu}\,(\mathbf{k}\,\boldsymbol{\Lambda} + \boldsymbol{\Lambda}\,\mathbf{k}) + \boldsymbol{\Lambda} \cdot \boldsymbol{\Lambda}\,\mathbf{E} + \frac{1}{1-2\nu}\,\boldsymbol{\Lambda}\,\boldsymbol{\Lambda}\right] \cdot \mathbf{a} = \mathbf{0}. \tag{6}$$

Let tensor $\mathbf{A}$ be the expression within square brackets in this equation. Nontrivial solutions for equation (6) exist if the determinant of tensor $\mathbf{A}$ equals 0. By evaluating the determinant we derive an equation to determine the characteristic values λ:

$$\det \mathbf{A} = \frac{2(1-2\nu)}{1-2\nu}\left(\lambda^2 - D^2\right)^3 = 0, \quad \text{where } D^2 = \boldsymbol{\Lambda} \cdot \boldsymbol{\Lambda} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{7}$$

This equation has two roots $\lambda = \pm D$, each of the multiplicity 3. Having done simple calculations we obtain the following representation for a solution of the initial value problem:

$$\mathbf{u} = \mathbf{P}(z, zD) \cdot \mathbf{f}(x, y) + \mathbf{Q}(z, zD) \cdot \mathbf{g}(x, y). \tag{8}$$

Tensors **P** and **Q** are to be considered as differential operators of the infinite order. They are analytical functions of operators D^2, $D \sin zD$, $\cos zD$. To write down the explicit form for tensors $\mathbf{P}, \mathbf{Q}$ we have to represent them in terms of series that include only integer powers of operator D^2, which is the two-dimensional Laplacian. Now, we have to derive the equations to determine functions $\mathbf{f}, \mathbf{g}$. We do this using the boundary conditions at $z = \pm h$. Finally, we obtain two vector equations with two-dimensional differential operators of the infinite order. Note that series for these operators converge very fastly. Therefore, usually it is possible to take into account only a few terms of the series. For instance, if we take into account only the principal term, we obtain equations of the classical Kirchoff theory of plates. The next approximation gives us the theory of plates taking into account a transverse displacement deformation. We shall not go into additional details here. However, we emphasize that the symbolic method described above has an absolutely rigorous mathematical proof. This is easy to generalize the method for the dynamical case, and it had been performed. The symbolic method of A.I. Lurie has had wide applications in the theory of thick plates, and it was used by many authors. The method turned to be the most effective in combination with the technique of homogeneous solutions, to which A.I. Lurie made a significant contribution as well.

The monograph "Theory of elasticity", near 650 pages, is beyond competition as for its fundamental nature and the scope of printed matter on the static problems of the elasticity. The dynamic problems and waves in elastic media are not considered in the book. The reason is not only the wish ot avoid the inevitable excessive increasement of the book size, but chiefly the fact that dynamic problems essentially differ from static ones by their physical and mathematical nature, and an inclusion of them into the work would destroy the integrity of description. It should be clearly realized that at the end of sixties, Russia held a far higher level relative to Science in comparison with 1925. There existed already a lot of exhaustive text-books for all areas of mechanics and scientific monographs being sometimes superior by their level to foreign analogues. The life had changed, and, in particular, the theoretical level of engineers had been essentially raised. A new kind of engineers, so called "engineers–researchers", had appeared. Their practical results were accompanied by rather deep theoretical investigations. The whole of the above-mentioned was taken into account by A.I. Lurie when he started to work on the monograph "Theory of elasticity". The problem was to give the exhaustive treatment of the subject, including the most important achievements of XIX–XX centuries in this field, basing on the same foundation. Naturally, the solution of this problem required the treatment of classical results in modern science language. In other words, there was a need for a thorough revision of printed matter in a huge quantity. In addition, the approaches to derivation of classical results, of course, were changed, in some cases essentially. At present, we can state that the monograph "Theory of elasticity" is in complete accordance to its designation. If anybody intends to gain the high class theoretical training in static problems of the elasticity theory, then the studying of "Theory of elasticity" is the shortest, although not too much easy, way to the aim. This is, no doubt, true with respect to the linear theory of elasticity. As for the section devoted to the nonlinear problems of the theory of elasticity, A.I. Lurie himself was not fully satisfied with the work performed[4]. A.I. Lurie's dissatisfaction with the treatment of the nonlinear problems in "Theory of elasticity" is easily explained. As it is known, the central problem in the nonlinear theory of elasticity is the formulation of the constitutive equations. In the linear theory of elasticity the constitutive equation is reduced, according to Cauchy's suggestion, to the general linear relation between the stress tensor and the strain tensor. In this case, the existence of the elastic potential is guaranteed. The only question arises when the restriction is imposed for the elastic potential to be positively defined. For an isotropic material the above restriction is reduced to the following inequalities:

$$\mu > 0, \qquad 3\lambda + 2\mu > 0, \tag{9}$$

where λ and μ are the Lamé constants. Are the restrictions (9) are necessary? The general laws of thermodynamics do not require these conditions to be true. They are not necessary from the formal mathematical point of view either. Indeed, the uniqueness and the existence of the solution for the equations of the nonlinear theory of elasticity is provided by the conditions of the strong ellipticity in statics or, which is just the same, by the conditions of the strong hyperbolicity in dynamics. The conditions lead to the necessity to fulfil the inequalities

$$\mu > 0, \qquad \lambda + 2\mu > 0, \tag{10}$$

which are more weak than the conditions (10). In the linear theory of elasticity the choice between inequalities (9) and (10) is made on the basis of the following physical principle: for any kind of the deformation of material from its natural state its internal energy, or, which is the same, its elastic potential must increase. The formulated principle was accepted in the mechanics of the elastic bodies as the stability concept. One can easily check that the material, which satisfies to the inequalities (10) with loss of the inequalities (9), can not exist continuously and it breaks spontaneously under action of arbitrary small loads. In the nonlinear theory of elasticity the situation is incomparably more complicated. Firstly, the elastic potential exists not for any relation between the stress tensor and the finite strain tensor. Therefore, it became necessary to distinguish the elastic and hyperelastic[5] materials. Secondly, the uniqueness of the solution of the static problem in the theory of elasticity is not only absent, but it must be absent. Thirdly, for the finite deformation the elastic potential does not necessarily increase along with the growth of deformation etc. In short, it is clear that the elastic potential could not be set in an arbitrary way and, at the same time, nobody knows what restrictions and why should be imposed on the elastic potential. As C. Truesdell proposed, this situation was named as a main unsolved problem of the theory of elasticity. By the time of publication of the monograph "Theory of elasticity", the problem mentioned above started to acquire the peculiar urgency. A new division of the nonlinear theory of elasticity started to form, which was named as "supplementary inequalities in the theory of elasticity". Within seventies, a variety of such inequalities was proposed, and the investigation of the consequences of violation of these inequalities started. For instance,

[4]By the way, many people told to the author of this communication that at the first acquaintance with the nonlinear theory of elasticity, they like far more the treatment of the theory in "Theory of elasticity" as compared to the much more extended monograph "Nonlinear theory of elasticity".

[5]The elastic potential exists for the latter in contrast to the former.

the problems where the condition of strong ellipticity is broken, became known as the singular problems in continuum mechanics. Certainly, A.I. Lurie could not stay aside of the questions discussed so intensively. However, in “Theory of elasticity” all these problems were not, and could not be elucidated. That is why, all at once, after the publication of “Theory of elasticity”, A.I. Lurie began to work on the new monograph “Nonlinear theory of elasticity”, which was published in six months after the author’s decease. The process of the working on the book was rather long, since it was needed not only to prepare the material for publication, but also to carry out an enormous research work at the fore and wide front of continuum mechanics. As always, A.I. Lurie had studied in detail all the latest advances of foreign scientists. For instance, he treated the reprint (1968) of lecture notes of C. Truesdell and recommended them for translation into Russian. This translation, named as “First Course in Rational Continuum Mechanics”, edited by A.I. Lurie, was published in 1975. In the course of translation A.I. Lurie was in active correspondence with C. Truesdell, who, as a result, made a lot of corrections and improvements to the initial text. Owing to this, the Russian translation of the book noticeably differed from the original. To the work on the problems described, A.I. Lurie drew his student E.L. Gurvich, with whom they published a joint paper “On the theory of wave propagation in the nonlinear elastic medium (an effective verification of the Adamar condition)”, Izvestia AN SSSR, MTT, 1980, N6, pp. 110–116. The paper was of a great importance in the theoretical sense. Unfortunately, A.I. Lurie was not fated to look through the paper in the published version. As we see, up to the end of his life, despite of the disease, which took a bad course after 1976, A.I. Lurie did not cease an active scientific work. The process of creation of the monograph “Nonlinear theory of elasticity” can not be treated as anything but the heroic scientific deed. It should be mentioned that the whole book, from the first to the last line, was written with the hand of A.I. Lurie. Nevertheless, only a very thoughtful reader could discover in it the signs of the severe illness, as traces of a certain hurry. A.I. Lurie clearly realized that his days are numbered, and he feared not be able to complete the ten-year work, which was of great importance for him. In considering the book as a whole, as all monographs by A.I. Lurie, it contains a thorough treatment of the subject in terms of direct tensor calculus, which essentially facilitates the perception of the material outlined. All presently used measures of deformation and stress tensors are introduced consequently. Contrary to the linear theory, one can introduce various stress and strain tensors in the nonlinear theory, and they must be distinguished rigorously. Naturally, the main attention is paid to the theory of constitutive equations and to the formulation of restrictions for these equations, and, in particular, the restrictions for setting the elastic potential. The problems for compressible and incompressible materials are considered in detail. Note that a rubber is an important example of an incompressible nonlinear elastic material. The variational principles of the nonlinear theory of elasticity are formulated in the monograph. In particular, one can find there the principle of complementary work, proved by L.M. Zubov, the student of A.I. Lurie. This principle initiated a spirited discussion in the foreign literature. A notable attention is given to such an important area of the nonlinear theory of elasticity, as a superposition of small and finite deformations. The importance of these problems is caused by the fact that in continuum mechanics, contrary to mechanics of systems with a finite number of degrees of freedom, the only way to examine the stability of the system is to consider a superposition of small and finite deformations. The great practical importance of stability problems is undoubtful. The monograph "Theory of nonlinear elasticity" by A.I. Lurie is closed with the statement of basic facts of thermodynamics for the nonlinear elastic medium.

7 Conclusion

Even from the above and rather brief review of A.I. Lurie's works on mechanics of solids and analytical mechanics, one can see how enormous his contribution into evolution of mechanics is. Moreover, the works of A.I. Lurie on the theory of control, where he obtained some world-famed results, are not discussed in this review. As for the contribution made by A.I. Lurie into evolution of mechanics in Russia, we can not pass over the School created by him, to which hundreds of students working in various fields of mechanics belong. The students continue the life-work of A.I. Lurie. The author of the communication had an honour not only to be a student — follower of the Teacher, but to spend many hours but to spend many hours together with him in a team work at his writting table. The most striking features of A.I. Lurie were his magnificent personal qualities, his perfect honesty and scientific uprightness along with kind and responsive regards for surrounding people. You should have seen how his eyes lit up with sincere interest and curiosity when discussing new scientific results obtained by someone! At the same time, he precisely distinguished at once a true new result from the known one but "got up in new clothes".

The name of A.I. Lurie has gone down in the history of Russian mechanics for ever.

Phase Transitions and General Theory of Elasto-Plastic Bodies*

Abstract

The paper deals with a new theory of elastoplastic bodies based on a description of inelastic properties by means of the phase transitions in the material. The medium is assumed to be micropolar. The theory is applicable to the materials in any phase states. Besides, the theory takes into account the dry friction between the particles of the medium.

1 Introduction

A behavior of solid structures under an external loading has been studied during several centuries. However intensive and task-oriented investigations had begun in XIX century and are carried out till the present time. All known materials can be separated on two different classes: elastic materials and all others. In general, the fundamentals of the nonlinear theory of elasticity may be considered as completely developed [1]. For inelastic materials the situation is quite different. There exists a huge massive of experimental data. This data is widely used for practical purposes and normative documents for the engineering projects, but as a rule this data is not used in theoretical investigations. A lot of established experimental facts cannot be described by the existing theories of the elastoplastic bodies till now. Let us point out some of them [2]: 1. Under sufficiently high pressure all materials experience irreversible strains (Bridgman), which can be considered as phase transitions. The rate of these transitions is determined by the properties of the material and do not depend on the rate of change of the external loads. 2. At sufficiently high pressure all rigid bodies flow similarly to a fluid (Tresca, Bridgman). For example, the classical experiment by H. Tresca on extrude of lead shows absence of the stagnant zones in the material. On contrary, from any existing theory of plasticity it follows that the bands of the “dead” material should be present [3]. Thus we see a serious qualitative discrepancy between the theory and experiment. 3. The experiments on large inelastic deformation show essential influence of the size effect [4]. 4. In all experiments with a smooth loading the Savart – Masson effect is exhibited clearly. 5. Experiments with bulk materials show the necessity of taking into account the dry friction between particles of the medium. All these facts are of great importance because they are observed practically in all experiments. Nevertheless, the existing theories of plasticity are

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not able to explain these facts. Moreover, the most of the known theories of plasticity are based on the yield criterions either by Saint-Venant – Tresca or by Mises. Both criterions were never strictly confirmed by experiments. While the existing theories cover almost all practical needs and extremely useful, nevertheless they are not able to explain some features of the material behavior.

The aim of the presentation is the attempt of build-up of such theory of inelastic materials, which would qualitatively feature the basic experimental facts. Besides, the theory should be sufficiently strict from the mathematical point of view. A novelty of the offered theory consists in the following. The experiments show that the inelastic materials cannot be modelled within framework of the material (Lagrangian) description. However the most of the known theories of the elastoplastic bodies are based on the material description. In what follows the spatial (Eulerian) description is used. The medium is assumed to be micropolar. Kinematics of the medium with rotational degrees of freedom is described. The fundamental laws are stated for open systems in a general form. The equation of the energy balance contains the term, which describes the formation of new particles or fragmentation of the initial particles. The concepts of internal energy, temperature and entropy are introduced by means of the pure mechanical arguments. The dry friction between the particles of the medium is introduced through the antisymmetric part of the stress tensor. The free energy is set in the form, which is suitable simultaneously for gases, fluids and solids. It is important to note that the material under the consideration has a finite tensile strength. That means that the constitutive equations can violate to the conditions of the strong ellipticity.

2 Fundamental Laws

2.1 Kinematic relations.

Let us consider a set of particles which are moving with respect to an inertial system. The set is not assumed to be a continuum. That means that the concept of a smooth differentiable manyfold cannot be used. Because of this a pure spatial (Eulerian) description will be used. Let a vector $\mathbf{V}(\mathbf{x}, t)$ be the velocity of a particle which at the actual instant of time t occupies the point $\mathbf{x}$ of a reference system. Let a quantity $\mathbf{K}(\mathbf{x}, t)$ be some property of the particle. In order to find the change of $\mathbf{K}(\mathbf{x}, t)$ during a motion of a particle we have to apply the material derivative [5]

$$\frac{\delta \mathbf{K}(\mathbf{x}, t)}{\delta t} \equiv \frac{d\mathbf{K}(\mathbf{x}, t)}{dt} + \left(\mathbf{V}(\mathbf{x}, t) - \frac{d\mathbf{x}}{dt}\right) \cdot \nabla \mathbf{K}(\mathbf{x}, t).$$

If the point $\mathbf{x}$ is moving with respect to the inertial reference system, then this definition does not coincide with conventional one and does not contradict with the Galilei principle of relativity. It is important to note that all used operators must be defined in the reference system rather than on smooth manyfold as at the material description. Besides let us point out that in the definition of a material derivative only the derivative $\mathbf{V} \cdot \nabla$ along the trajectory of a particle is used. Thus the continuity of $\mathbf{K}(\mathbf{x}, t)$ with respect to the space variable $\mathbf{x}$ is not assumed. For a vector of the particle acceleration we have

$$\mathbf{W}(\mathbf{x}, t) = \frac{d}{dt}\mathbf{V}(\mathbf{x}, t) + \left(\mathbf{V}(\mathbf{x}, t) - \frac{d\mathbf{x}}{dt}\right) \cdot \nabla \mathbf{V}(\mathbf{x}, t).$$

Let us introduce the displacement vector

$$\frac{\delta \mathbf{u}(\mathbf{x},t)}{\delta t} = \mathbf{V}(\mathbf{x},t) \;\Rightarrow\; \frac{d\mathbf{u}}{dt} = \mathbf{V}\cdot\mathbf{g}, \tag{1}$$

where

$$\mathbf{g}(\mathbf{x},t) \equiv (\mathbf{E} - \nabla\mathbf{u}(\mathbf{x},t)), \quad \det\mathbf{g}(\mathbf{x},t) > 0. \tag{2}$$

The tensor $\mathbf{g}(\mathbf{x},t)$ will be termed the first measure of deformation. The Eq.(1) is a definition of the displacement vector. From (1) it follows

$$\nabla\mathbf{V}(\mathbf{x},t) = -\frac{\delta\mathbf{g}(\mathbf{x},t)}{\delta t}\cdot\mathbf{g}^{-1}(\mathbf{x},t). \tag{3}$$

Eqs.(1)–(3) may be found in [6] and will be used in the reduced equation of the energy balance. If a tensor $\mathbf{P}(\mathbf{x},t)$ determines the rotation of a particle, then the angular velocity of the particle is defined by the modified Poisson equation [5]

$$\frac{\delta\mathbf{P}(\mathbf{x},t)}{\delta t} = \boldsymbol{\omega}(\mathbf{x},t)\times\mathbf{P}(\mathbf{x},t). \tag{4}$$

Let us introduce the second measure of deformation $\mathbf{F}$ by means of equalities

$$\frac{\partial}{\partial x^s}\mathbf{P} = \mathbf{F}_s\times\mathbf{P}, \quad \mathbf{F} = \mathbf{g}^s\otimes\mathbf{F}_s, \tag{5}$$

where the vectors $\mathbf{g}_s$ are the basis vectors and the following conditions of integrability hold [5]

$$\frac{\partial\mathbf{F}_s}{\partial x^m} - \frac{\partial\mathbf{F}_m}{\partial x^s} = \mathbf{F}_m\times\mathbf{F}_s. \tag{6}$$

From Eq.(6) it follows

$$\nabla\otimes\boldsymbol{\omega} = \frac{\delta\mathbf{F}}{\delta t} + \mathbf{F}\times\boldsymbol{\omega} + \nabla\mathbf{V}\cdot\mathbf{F}.$$

2.2 Particles and mass balance.

Let us introduce two nonnegative functions: $\eta(\mathbf{x},t)$ is the particle density and $\rho(\mathbf{x},t)$ is the mass density. If the material has a tendency to a fragmentation, then the total mass is conserved, but the number of particles does not conserved. In such a case the following equations are valid

$$\frac{\delta\eta}{\delta t} + \eta\nabla\cdot\mathbf{V} = \chi, \quad \frac{\delta\rho}{\delta t} + \rho\nabla\cdot\mathbf{V} = 0, \tag{7}$$

where $\chi(\mathbf{x},t)$ determines the production of new particles for the unit of time. From practical point of view the importance of $\eta(\mathbf{x},t)$ is determined by the necessity to take into account the porosity of material. In such a case the function $\chi(\mathbf{x},t)$ in Eqs.(7) depends on pressure. Using the identity [5]

$$\nabla\cdot\mathbf{V}(\mathbf{x},t) = -\frac{1}{I_3(\mathbf{g})}\frac{\delta I_3(\mathbf{g})}{\delta t}, \quad I_3(\mathbf{g}) \equiv \det\mathbf{g}$$

the above equations can be written in the form

$$\frac{\delta}{\delta t}\left[\frac{\eta}{I_3(\mathbf{g})}\right]=\frac{\chi}{I_3}, \quad \frac{\delta}{\delta t}\left[\frac{\rho}{I_3(\mathbf{g})}\right]=0. \tag{8}$$

Let us introduce a some characteristic of a particle called the particle volume ν_p. The quantity $c=\nu_p\eta$, known as the compactness, determines the material volume $\nu_p\eta dV$ occupied by the material in the control volume dV. The quantity $c_p=1-\nu_p\eta$ is termed a porosity. Note that we apply the term "porosity" in unconventional sense because we do not consider the porous medium. We mean that any solid material has a several stable states corresponding to different magnitudes of the compactness. The transition of the material from one stable state to another stable state is a typical phase transition which we would like to take into account. For all known materials compactness satisfies an inequality $0 \leq \nu_p\eta \leq 0.74$. Thus for porosity we have $0.26 \leq c_p \leq 1$. The first equation from Eqs.(7) may be rewritten in terms of porosity

$$\frac{\delta c_p}{\delta t}+\nu_p\chi(c_p,\, p)=\boldsymbol{\nabla}\cdot\mathbf{V}, \tag{9}$$

where p is a pressure and the function $\nu_p\chi(c_p,\, p)$ must be defined by the constitutive equation, for which there exist a many different possibilities, but the final results are not known. Because of this we are not able to give a short resume of these possibilities. As an example the following equation may be considered

$$\frac{dc_p}{dt}+\mathbf{V}\cdot\boldsymbol{\nabla}c_p=\boldsymbol{\nabla}\cdot\mathbf{V}-\frac{\epsilon^2 p}{\epsilon^2+(p_c-p)^2},$$

where $\epsilon^2 \ll 1$ is a small parameter and p_c is some critical pressure. This equation shows the behavior of the porosity near one point of the phase transitions. The realistic equations should have a more complicated form.

2.3 Dynamics Laws.

Let us introduce the stress tensor $\mathbf{T}(\mathbf{x},t)$ and the moment stress tensor $\mathbf{M}(\mathbf{x},t)$. These tensors are defined in the space, but not in the material. For them the Cauchy formulae are valid

$$\mathbf{T}_{(n)}=\mathbf{n}\cdot\mathbf{T}, \quad \mathbf{M}_{(n)}=\mathbf{n}\cdot\mathbf{M}.$$

The first and the second laws by Euler have the well known form

$$\boldsymbol{\nabla}\cdot\mathbf{T}+\rho\mathbf{F}=\rho\frac{\delta\mathbf{V}(\mathbf{x},t)}{\delta t}, \tag{10}$$

$$\boldsymbol{\nabla}\cdot\mathbf{M}+\mathbf{T}_{\times}+\rho\mathbf{L}=\rho\frac{\delta(\mathbf{J}\cdot\boldsymbol{\omega})}{\delta t}, \tag{11}$$

where the mass density of the inertia tensor $\mathbf{J}$ of a particle in the actual position is connected with the constant tensor $\mathbf{J}_0$ in the reference position by

$$\mathbf{J}(\mathbf{x},t)=\mathbf{P}(\mathbf{x},t)\cdot\mathbf{J}_0\cdot\mathbf{P}^T(\mathbf{x},t). \tag{12}$$

2.4 Equation of the energy balance.

The equation of the energy balance in the local form can be written down as

$$\rho\frac{\delta\mathcal{U}}{\delta t} = \mathbf{T}^{\mathsf{T}} \cdot\cdot \left(\nabla\mathbf{V} + \mathbf{E} \times \boldsymbol{\omega} \right) + \mathbf{M}^{\mathsf{T}} \cdot\cdot \nabla\, \boldsymbol{\omega} + \nabla\cdot \mathbf{h} + \rho q. \tag{13}$$

where $\mathcal{U}$ is the mass density of internal energy and the vector **h** is the vector of the heat flux. The right hand side of Eq.(13) contain the power of the stress tensor and of the moment stress tensor. One part of the power changes the specific internal energy. Another part partly remains in the body as a heat and partly radiates into the external medium. In order to separate these parts the stress tensor and the moment stress tensor must be represented as

$$\mathbf{T} = \mathbf{T}_e + \mathbf{T}_i, \quad \mathbf{M} = \mathbf{M}_e + \mathbf{M}_i, \tag{14}$$

where the quantities with the subscript "e" are independent of velocities and the quantities with the subscript "i" are the rest part of stresses. One may substitute Eq.(14) into Eq.(13) in order to get the final form of the energy balance equation. However in such a form the energy balance equation is almost useless. We have to transform this equation in order to obtain the reduced equation of the energy balance.

The forth fundamental law of mechanics is the second law of thermodynamics. The statement of this law will be given in the following section.

3 The heat conductivity equation

Let us introduce the concepts of the temperature, entropy and chemical potential by means of the following equation

$$\rho\,\vartheta\,\frac{\delta\mathcal{H}}{\delta t} + \rho\,\eta\,\frac{\delta\mathcal{C}}{\delta t} = \nabla\cdot\mathbf{h} + \rho q + \mathbf{T}_i^{\mathsf{T}} \cdot\cdot \left(\nabla\mathbf{V} + \mathbf{E} \times \boldsymbol{\omega} \right) + \mathbf{M}_i^{\mathsf{T}} \cdot\cdot \nabla\boldsymbol{\omega}, \tag{15}$$

where the functions $\vartheta(\mathbf{x}, t)$, $\mathcal{H}(\mathbf{x}, t)$ and $\mathcal{C}(\mathbf{x}, t)$ are respectively termed the temperature, the specific entropy and the specific chemical potential. Let us underline that Eq.(15) is the definition for these functions. The only purpose of introduction of the specific entropy and the specific chemical potential (these functions by itself have no physical sense) is to define by an appropriate way the temperature ϑ and the particle density η or, what is the same, porosity of the material. Of course, we need some additional assumptions for a complete definition of those quantities. Now let us accept the second law of thermodynamics in the form of the following inequalities

$$\mathbf{T}_i^{\mathsf{T}} \cdot\cdot \left(\nabla\mathbf{V} + \mathbf{E} \times \boldsymbol{\omega} \right) + \mathbf{M}_i^{\mathsf{T}} \cdot\cdot \nabla\boldsymbol{\omega} \geq 0, \quad \mathbf{h} \cdot \nabla\vartheta \geq 0. \tag{16}$$

Inequalities (16) are more strong than the consequences of the known inequality by Clausius – Duhem [7]. However from our point of view Eq.(16) are quite good for practical aims. The constitutive equation for the vector of the heat flux may be taken in the simplest form

$$\mathbf{h} = -\,\kappa\,\nabla\vartheta, \quad \kappa \geq 0. \tag{17}$$

The substituting of Eq.(17) into Eq.(15) leads to the heat conduction equation.

4 Reduced equation of the energy balance, the Cauchy – Green relations

Let us introduce the specific free energy

$$\mathcal{F} = \mathcal{U} - \vartheta\mathcal{H} - \eta\mathcal{C}. \tag{18}$$

Making use Eqs.(14), (15) and (18) the equation (13) can be rewritten in the following form

$$\rho\frac{\delta\mathcal{F}}{\delta t} + \rho\mathcal{H}\frac{\delta\vartheta}{\delta t} + \rho\mathcal{C}\frac{\delta\eta}{\delta t} = \mathbf{M}_e^\top\cdot\cdot\frac{\delta\mathbf{F}}{\delta t} - \left(\mathbf{g}^{-1}\cdot\mathbf{T}_e^\top + \mathbf{g}^{-1}\cdot\mathbf{F}\cdot\mathbf{M}_e^\top\right)\cdot\cdot\frac{\delta\mathbf{g}}{\delta t} + \\ + \frac{1}{2}\left[\left(\mathbf{M}_e^\top\cdot\mathbf{F} - \mathbf{T}_e\right)_\times \times \mathbf{P}\right]^\top\cdot\cdot\frac{\delta\mathbf{P}}{\delta t}. \tag{19}$$

The equation of the energy balance written in the form like Eq.(19) is termed the reduced equation of the energy balance. This equation involves only the intrinsic variables. From Eq.(19) we see that the free energy is a function of the following arguments

$$\mathcal{F} = \mathcal{F}\left(\vartheta, \eta, \mathbf{g}, \mathbf{F}, \mathbf{P}\right). \tag{20}$$

Taking into account this statement it is readily to derive the Cauchy–Green relations

$$\mathcal{H} = -\frac{\partial\mathcal{F}}{\partial\vartheta}, \quad \mathcal{C} = -\frac{\partial\mathcal{F}}{\partial\eta}, \quad \mathbf{M}_e = \rho\,\frac{\partial\mathcal{F}}{\partial\mathbf{F}}, \quad \mathbf{T}_e = -\rho\,\frac{\partial\mathcal{F}}{\partial\mathbf{F}}\cdot\mathbf{F}^\top - \rho\,\frac{\partial\mathcal{F}}{\partial\mathbf{g}}\cdot\mathbf{g}^\top. \tag{21}$$

Now Eq.(20) takes the form

$$\rho\left(\frac{\partial\mathcal{F}}{\partial\mathbf{P}}\right)^\top\cdot\cdot\frac{\delta\mathbf{P}}{\delta t} = \frac{1}{2}\left(\left(\mathbf{M}^\top\cdot\mathbf{F} - \mathbf{T}\right)_\times \times \mathbf{P}\right)^\top\cdot\cdot\frac{\delta\mathbf{P}}{\delta t}. \tag{22}$$

Here we have to take into account that the material derivative of the tensor $\mathbf{P}$ cannot be changed by an arbitrarily manner. Indeed, from the Poisson equation Eq.(4) it follows

$$\frac{\delta\mathbf{P}(\mathbf{x},t)}{\delta t}\cdot\mathbf{P}^\top(\mathbf{x},t) = \boldsymbol{\omega}(\mathbf{x},t)\times\mathbf{E} \quad \Rightarrow \quad (\mathbf{A}\cdot\mathbf{P})^\top\cdot\cdot\frac{\delta\mathbf{P}(\mathbf{x},t)}{\delta t} = 0, \quad \forall\mathbf{A} : \mathbf{A} = \mathbf{A}^\top.$$

Hence we get the relation

$$\rho\,\frac{\partial\mathcal{F}}{\partial\mathbf{P}} - \frac{1}{2}\left(\mathbf{M}_e^\top\cdot\mathbf{F} - \mathbf{T}_e\right)_\times \times \mathbf{P} = \mathbf{A}\cdot\mathbf{P}.$$

In order to exclude the arbitrary symmetric tensor $\mathbf{A}$, we have to multiply both sides of this equation by the tensor $\mathbf{P}^\top$ and to calculate the vector invariants of both sides. As a result we have

$$\left[\rho\,\frac{\partial\mathcal{F}}{\partial\mathbf{P}}\cdot\mathbf{P}^\top + \mathbf{M}_e^\top\cdot\mathbf{F} - \mathbf{T}_e\right]\cdot\cdot\mathbf{C} = 0, \quad \forall\,\mathbf{C} :\ \mathbf{C} = -\mathbf{C}^\top. \tag{23}$$

The stress tensor $\mathbf{T}_e$ and the moment stress tensor $\mathbf{M}_e$ are defined by the Cauchy–Green relations Eqs.(21). That means that the condition Eq.(23) is the restriction superposed on the free energy.

Below we use the technics given in [8]. From Eq.(23) we see that the free energy must satisfy the following equation of first order partial differential equation

$$\left(\frac{\partial \mathcal{F}}{\partial \mathbf{g}}\right)^{T} \cdot\cdot\, (\mathbf{C}\cdot\mathbf{g}) + \left(\frac{\partial \mathcal{F}}{\partial \mathbf{P}}\right)^{T} \cdot\cdot\, (\mathbf{C}\cdot\mathbf{P}) + \left(\frac{\partial \mathcal{F}}{\partial \mathbf{F}}\right)^{T} \cdot\cdot\, (\mathbf{C}\cdot\mathbf{F} - \mathbf{F}\cdot\mathbf{C}) = 0. \qquad (24)$$

The characteristic system for Eq.(24) has a form

$$\frac{d\mathbf{g}}{ds} = \mathbf{C}\cdot\mathbf{g}, \quad \frac{d\mathbf{P}}{ds} = \mathbf{C}\cdot\mathbf{P}, \quad \frac{d\mathbf{F}}{ds} = \mathbf{C}\cdot\mathbf{F} - \mathbf{F}\cdot\mathbf{C}. \qquad (25)$$

The free energy to satisfy Eq.(24) must be a function of the integrals of Eq.(25). The latter consists the system of the order 21 and has not more than 18 functionally independent integrals of Eq.(25).

5 Nonpolar medium with the Coulomb friction

Let us assume that the free energy is independent of the second deformation measure **F**

$$\mathcal{F} = \mathcal{F}\left(\vartheta,\, \eta,\, \mathbf{g}\right), \quad \mathbf{M}_e = \mathbf{0}. \qquad (26)$$

We may rewrite Eq.(19) as

$$\rho\frac{\delta\mathcal{F}}{\delta t} + \rho\mathcal{H}\frac{\delta\vartheta}{\delta t} + \rho\mathcal{C}\frac{\delta\eta}{\delta t} = -\left(\mathbf{g}^{-1}\cdot\mathbf{T}_e^{T}\right)\cdot\cdot\frac{\delta\mathbf{g}}{\delta t}.$$

The stress tensor can be decomposed as

$$\mathbf{T}_e = -p\,\mathbf{E} + \boldsymbol{\tau}, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{T}, \quad \mathrm{tr}\,\boldsymbol{\tau} = 0.$$

The representations for $\mathbf{T}_i$ and $\mathbf{M}_i$ will be given below. Making use the technics given in the previous section one can prove that in case under consideration the free energy has a form

$$\mathcal{F} = \mathcal{F}\left(\vartheta,\, \eta,\, \rho,\, \gamma,\, \mathbf{G}\right),$$

where

$$\gamma \equiv I_3^2(\mathbf{g}), \quad \mathbf{G} \equiv I_3^{-2/3}\mathbf{g}^{T}\cdot\mathbf{g}, \quad \det\mathbf{G} = 1.$$

Following [6] the unimodular tensor **G** will be termed the strain of shape change. The constitutive equations for the pressure p and for the deviator $\boldsymbol{\tau}$ take a form

$$p = \rho^2\frac{\partial\mathcal{F}}{\partial\rho} + \rho I_3(\mathbf{g})\frac{\partial\mathcal{F}}{\partial I_3(\mathbf{g})}, \quad \boldsymbol{\tau} = -2\rho\left[\gamma^{-1/3}\mathbf{g}\cdot\frac{\partial\mathcal{F}}{\partial\mathbf{G}}\cdot\mathbf{g}^{T} - \frac{1}{3}\mathbf{G}\cdot\cdot\frac{\partial\mathcal{F}}{\partial\mathbf{G}}\mathbf{E}\right].$$

Let us introduce the new parameters

$$\zeta = \frac{1}{\sqrt{\rho I_3(\mathbf{g})}}, \quad z = \sqrt{\frac{\rho}{I_3(\mathbf{g})}}, \quad \frac{\delta z}{\delta t} = 0.$$

In such a case we have the final form of constitutive equations

$$\mathcal{H} = -\frac{\partial \mathcal{F}}{\partial \vartheta}, \quad \mathcal{C} = -\frac{\partial \mathcal{F}}{\partial \eta}, \quad p = -\frac{\partial z\mathcal{F}}{\partial \zeta}, \quad -\frac{\zeta \boldsymbol{\tau}}{2} = \gamma^{-1/3}\mathbf{g} \cdot \frac{\partial z\mathcal{F}}{\partial \mathbf{G}} \cdot \mathbf{g}^{T} - \frac{1}{3}\mathbf{G} \cdot\cdot \frac{\partial z\mathcal{F}}{\partial \mathbf{G}}\mathbf{E}, \quad (27)$$

where the free energy is a function of five arguments

$$\mathcal{F} = \mathcal{F}(\vartheta, \eta, \zeta, z, \mathbf{G}). \quad (28)$$

Now let us assume the following representations for the viscous stresses

$$\mathbf{T}_i = \mathbf{t} \times \mathbf{E}, \quad \mathbf{M}_i = \mathbf{m} \times \mathbf{E}. \quad (29)$$

With these assumptions the first inequality from Eq.(16) takes the form

$$-\mathbf{t} \cdot (2\boldsymbol{\omega} - \boldsymbol{\nabla} \times \mathbf{V}) - \mathbf{m} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\omega}) > 0.$$

For the moment vector **m** we take the viscous friction law and for the stress vector **t** we assume that the Coulomb dry friction law is valid

$$\mathbf{t} = -k\, h(\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n})\, |\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n}| \frac{(2\boldsymbol{\omega} - \boldsymbol{\nabla} \times \mathbf{V})}{|2\boldsymbol{\omega} - \boldsymbol{\nabla} \times \mathbf{V}|}, \quad \mathbf{m} = -\mu_m (\boldsymbol{\nabla} \times \boldsymbol{\omega}), \ \mu_m \geq 0, \quad (30)$$

where the function $h(\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n})$ is determined by

$$h(\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n}) = \begin{cases} 1, & \mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n} < 0, \\ 0, & \mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n} \geq 0, \end{cases}$$

$k \geq 0$ is the coefficient of the dry friction. The unit vector **n** in Eq.(30) must be found as a solution of the problem

$$\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{m} = \max, \quad \forall\, \mathbf{n}, \mathbf{m} : |\mathbf{n}| = |\mathbf{m}| = 1, \quad \mathbf{n} \cdot \mathbf{m} = 0.$$

It is easy to prove that the solution of this problem is unique. The Coulomb law in Eq.(30) is applicable if a sliding is present. Otherwise we have a condition

$$2\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{V}, \quad (31)$$

and the vector **t** has to be found from Eq.(11)

$$-\mu_m \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{V})] - 4\mathbf{t} = \rho \frac{\delta}{\delta t} [\mathbf{J} \cdot (\boldsymbol{\nabla} \times \mathbf{V})]. \quad (32)$$

Using Eq.(32) the vector **t** can be eliminated from the first law of dynamics.

6 Isotropic materials

Let us suppose that we deal with isotropic materials. In such a case the free energy depends on the invariants of the tensor **G**

$$\mathcal{F} = \mathcal{F}(\vartheta, \eta, \zeta, z, I_1, I_2), \quad I_1(\mathbf{G}) \equiv \mathbf{E} \cdot\cdot \mathbf{G}, \quad I_2(\mathbf{G}) \equiv \mathbf{G} \cdot\cdot \mathbf{G}. \quad (33)$$

Making use of Eq.(33) we can rewrite Eqs.(20) as

$$\mathcal{H} = -\frac{\partial \mathcal{F}}{\partial \vartheta}, \quad \mathcal{C} = -\frac{\partial \mathcal{F}}{\partial \eta}, \quad p = -\frac{\partial z\mathcal{F}}{\partial \zeta},$$

$$\zeta \boldsymbol{\tau} = \frac{2}{3}\left(I_1 \frac{\partial z\mathcal{F}}{\partial I_1} + I_2 \frac{\partial z\mathcal{F}}{\partial I_2} \right)\mathbf{E} - 2\left(\frac{\partial z\mathcal{F}}{\partial I_1}\boldsymbol{\Lambda} + \frac{\partial z\mathcal{F}}{\partial I_2}\boldsymbol{\Lambda}^2 \right), \quad (34)$$

where the tensor $\boldsymbol{\Lambda}$ is defined by

$$\boldsymbol{\Lambda} = I_3^{-2/3}(\mathbf{g})\,\mathbf{g}\cdot\mathbf{g}^T.$$

The invariants of the tensor $\boldsymbol{\Lambda}$ are given by

$$I_1(\mathbf{G}) = I_1(\boldsymbol{\Lambda}) = \Lambda_1 + \Lambda_2 + \frac{1}{\Lambda_1\Lambda_2} \geq 3, \quad I_2(\mathbf{G}) = I_2(\boldsymbol{\Lambda}) = \Lambda_1^2 + \Lambda_2^2 + \frac{1}{\Lambda_1^2\Lambda_2^2} \geq 3,$$

where Λ_1, Λ_2 are two independent eigenvalues of $\boldsymbol{\Lambda}$.

7 Constitutive equation for the pressure

Let us assume that the free energy may be represented as a composition

$$z\mathcal{F} = f(\vartheta, \eta, \zeta, z) + z\mathcal{F}_d(\vartheta, \eta, z, I_1, I_2).$$

In such a case the pressure is determined by

$$p = -\frac{\partial f}{\partial \zeta}. \quad (35)$$

Let the pressure p be a linear function of the temperature

$$p = f_1(\zeta, \eta, z) + f_2(\zeta, \eta, z)\,\vartheta. \quad (36)$$

The most popular in physics of solids the constitutive equations by van-der-Waals and by Mu–Grüneisen have namely this form. For example, the van-der-Waals equation can be written as

$$p(\zeta, \vartheta) = -\frac{a}{\zeta^2} + \frac{c\,\vartheta}{\zeta - b}, \quad (37)$$

where a, b and c are the characteristics of the material. However, in our case these quantities may depend on the parameters η, z. It is known that the van-der-Waals equation satisfactory predicts the behavior of the real gases. It seems obvious that Eq.(36) can be by corresponding choice of the functions f_1 f_2 used not only for liquids and gases but for solids with the phase transitions. The pressure at $\vartheta = 0$ is described by means of the function $f_1(\zeta, \eta, z)$, the possible form of which is shown in Figure 1. The material shown in Figure 1 has three stable equilibrium states. The transition from one state to another is a typical phase transition. It is easy to understand that the diagram, like shown in Figure 1, cannot be found by experiment. However, the envelops of the true diagram can be established in an experiment. The upper envelope describes the properties of a material under compression, and the lower envelope describes the properties of the material

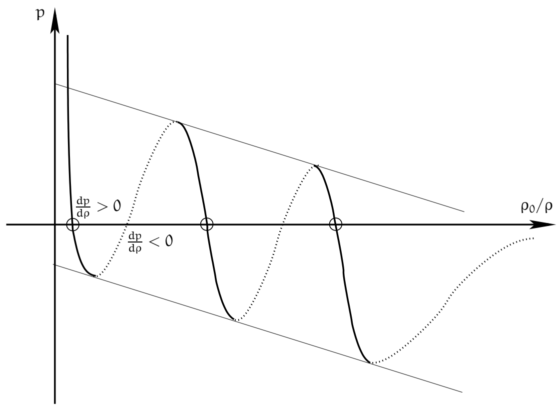

Figure 1: Constitutive equation for the pressure at zero temperature.

under extension. At the qualitative consideration the function $f_2(\zeta, \eta, z)$ may be chosen as in the van-der-Waals equation. The simplest example of the constitutive equation for the material with finite tensile strength is given by the expression

$$p = f_0 \left(\zeta^{-m} - \zeta^{-n}\right) + \frac{c\,\vartheta}{\zeta - b}, \tag{38}$$

where $m > n$, $\zeta > b(\eta)$. The pressure dependence on ζ at different temperatures is shown in Figure 2. Using Eq.(36) and Eq.(38) we find the expression for the free energy

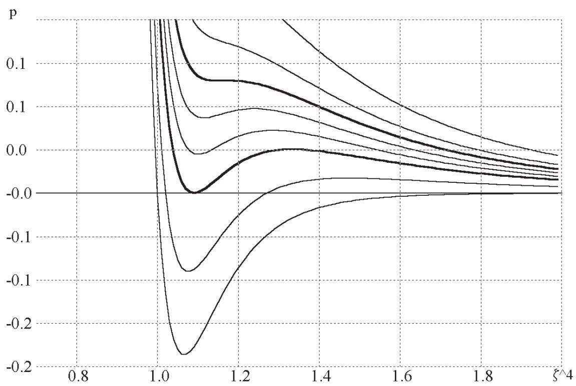

Figure 2: The pressure dependence on ζ^4 for different temperatures.

$$\rho_0\, \mathcal{F}(\zeta, \vartheta, \mathbf{E}) = f_0 \left(\frac{\zeta^{-m+1}}{m-1} - \frac{\zeta^{-n+1}}{n-1}\right) - c\,\vartheta\, \ln(\zeta - b) + \psi(\vartheta),$$

where $\psi(\vartheta)$ is some function of temperature. More general form of the constitutive equation for the pressure is given by

$$p = \sum_{k=0}^{N} a_k \zeta^{-k} + \frac{c\,\vartheta}{\zeta - b}, \tag{39}$$

where parameters (a_k, N, c, b) are characteristics of material. All of them may depend on the structural parameter η. Besides, maybe it will be useful to take more general form of the function f_2. In general, Eq.(39) corresponds to the material with N solid phase states. If we desire to take into account the phase transition "solid–liquid" and "liquid–gas", then we have to construct the curve like shown in Figure 3. If it is desirable to take into account a several solid phase states, then we have to add to the constitutive equation the terms like the first term in the right hand side of Eq.(39).

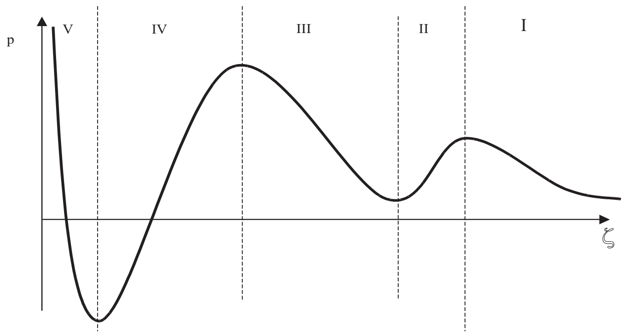

Figure 3: The three-phases medium: zones I, III, V correspond to stable gaseous, liquid and solid phases respectively; zones II, IV correspond the unstable states

8 Constitutive equation for the stress tensor deviator

From conventional point of view the state equation of solid is the relation between pressure, temperature and the mass density or volume. However the constitutive equation for the deviator $\boldsymbol{\tau}$ of the stress tensor cannot be ignored. Let us underline that the most of the phase transitions in solid are connected with the fact that the maximal shear stress in material has a rather low upper limit. When defining the function zF_d we have first of all to take into account this fact. Let the values $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$ $(\lambda_1\lambda_2\lambda_3 = 1)$ be eigenvalues of the tensor $\mathbf{G}$. Let us introduce the quantity

$$\sigma \equiv 3I_2(\mathbf{G}) - I_1^2(\mathbf{G}) = (\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2 + (\lambda_2 - \lambda_3)^2. \tag{40}$$

If $\sigma = 0$, then $\mathbf{G} = \mathbf{E}$. Now let us assume that the deviatoric part of free energy $z\mathcal{F}_d$ depends on the parameters σ and I_1 rather than invariants I_1, I_2. In such a case from Eqs.(34) one may obtain the equation

$$\zeta\boldsymbol{\tau} = -\frac{\partial z\mathcal{F}}{\partial I_1}\left(\boldsymbol{\Lambda} - \frac{1}{3}I_1\mathbf{E}\right) + 2\frac{\partial z\mathcal{F}}{\partial \sigma}\left[\frac{(\sigma - I_1^2)}{3}\mathbf{E} + 2I_1\boldsymbol{\Lambda} - 3\boldsymbol{\Lambda}^2\right]. \tag{41}$$

If we consider the case of small deformations when $||(\nabla \mathbf{u}|| \ll 1$, then instead of Eq.(41) we get

$$\zeta\boldsymbol{\tau} = 2\mu\, \mathrm{dev}\varepsilon + O(\varepsilon^2), \quad \mu \equiv \frac{\partial z\mathcal{F}}{\partial I_1}, \tag{42}$$

where ε is the tensor of linear deformations, the parameter μ may be termed the shear modulus. From Eq.(42) we see that in linear theory the dependence of free energy on a parameter σ is not important. By this reason and for the sake of simplicity we assume that the free energy does not depend on the parameter σ and Eq.(41) takes a form

$$\zeta\boldsymbol{\tau} = 2\mu\left(\frac{1}{3}I_1\mathbf{E} - \boldsymbol{\Lambda}\right), \quad \mu \equiv \frac{\partial z\mathcal{F}}{\partial I_1}. \tag{43}$$

The shear modulus μ is a function like $\mu = \mu(\vartheta, \eta, z, I_1)$. In order to define the function $\mu(\vartheta, \eta, z, I_1)$ we have many possibilities. But at the moment it is difficult to understand by unique manner what possibility is used by the Nature. The shear modulus depends on four different parameters ϑ, η, z, I_1. For all of them we have governing equations. We thing that the dependence μ from the temperature is not crucial one. The same may be said with respect to the variable z. However, the parameters η and I_1 have a crucial influence on the shear modulus. The problem is that from physical point of view both η and I_1 influences on the shear modulus in the almost similar way. As far as we know in mechanics of solids the parameter η has never been used and the behavior of the shear modulus is determined by deformations. In such a case it is possible to use, for example, the following representation

$$\mu = \mu_0(\vartheta, z)\left[1 - \cos\left(\frac{\pi(I_1 - 3)}{2l_*}\right)\right] \tag{44}$$

where l_* is a some characteristic of the material. The representation Eq.(44) corresponds to the free energy which looks just like the potential by Frenkel – Kontorova [4] in dynamics of crystal lattice. We do not think that this representation is sufficiently good for practical needs. At the moment we would like to point out the qualitative behavior of the shear modulus. We have to remember that under high pressure the shear modulus must vanish. The dependence of shear modulus on I_1 is not monotone in order to describe the Savart – Masson effect.
Maybe, more realistic constitutive equation for shear modulus is given by representation like

$$\mu = \mu_0(\vartheta, z, I_1)(1 - c_p)^2(c_p - 0.26)^2, \tag{45}$$

where c_p is the porosity which must satisfy Eq.(9). Small values of $(1 - c_p)$ occur for the gases. For solids c_p is close to 0.26. The quantitative dependence of μ from c_p may be, of course, different from Eq.(45). It is quite possible that we will need some combination of the representations like Eq.(44) and Eq.(45). The future investigations have to clear the situation.

Conclusion

Above a general (maybe, superfluous general) theory of materials in any phase states is developed. The present state of the theory does not suit for those people who desire to obtain the practical results immediately. But what do we know? We know that during more than 150 years the applied theories of inelastic materials were developed in great extent. And in spite of this there exist a lot of very old experimental results which cannot be described by the existing theories. Why? Maybe, it is time to go far from practical results and to develop the theory which is right from fundamental point of view. We think that any applied theory must be consistent with the theory of such a kind. At the moment we have made only initial steps. However even from these initial steps we see that some conventional statements are not valid. For example, everybody knows that $I_3(\mathbf{g})$ is responsible for a volume change of a material. But we saw that it is not so and we have to introduce the special object to characterize the material volume. From physical point of view it is clear that the important role in the description of inelastic properties of the material must play the chemical potential which is responsible for the structural transformation in the material. As far we know the chemical potential was used in continuum mechanics only in the case of the multi-component media.

We hope that the given above theory attracts the attention both physicists and mathematicians. The theory is needed in additional minds to create the true useful applied theory.

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Generalized Continuum and Linear Theory of Piezoelectric Materials*

Abstract

Theory of the piezoelectric materials had been developed many years ago. It was supposed that the stress state of the piezoelectric material can be described by means of the symmetrical stress tensor. However it can be shown that as a matter of fact the particles of the piezoelectric material must be considered as dipoles. It means that the theory of the piezoelectric materials must be constructed on the base of the generalized continuum. The theory of such a kind is presented in the report. The basic equations are derived from the fundamental laws of Eulerian mechanics. It is shown that the type of the electric field vector is important, since it influences on the structure of the basic equations.

1 Introduction

The brothers Pierre and Jacques Curie, in 1880, were the first to experimentally demonstrate piezoelectric behavior in a series of crystals, including quartz and Rochelle salt. The first attempt to derive the theory of piezoelectricity was made by Voigt in 1910.

Crystals with piezoelectric properties are very useful for different scientific and industrial applications. The direct piezoelectric effect occurs when an applied stress produces an electric polarization. The inverse piezoelectric effect occurs when an applied electric field produces a strain. These coupled effects let the electronic industry to produce many useful devices such as piezoelectric crystals, filters and resonators. First crystals were created by W. Cady in 1923 on the base of the natural α-quartz. To the present time the construction and characteristics of crystals were essentially improved.

There exist a several theories of the piezoelectricity. All of them lead to the very complicated equations. The exact solutions of these equations may be found only for very particular cases. By this reason it is not easy to compare theoretical and experimental results. At the present time it seems to be possible to say that there is no qualitative discrepancies between theory and experiments. From the pure theoretical point of view in the theory of the piezoelectricity there are some serious problems. The first problem. In electrodynamics the choice of the type of the electric field vector **E** does not matter and there are no reasons to make this choice. In the

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piezoelectricity it is not so and the type of $\mathbf{E}$ is important. In conventional theories the vector $\mathbf{E}$ is supposed to be a polar vector. In what follows we consider the theories when the type of $\mathbf{E}$ may be changed. The second problem. At least some piezoelectric materials are the dipole crystals. In such a case the rotation degrees of freedom must be taken into account.

2 The classical theory of piezoelectricity

There are several theories [1, 2, 3] to describe piezoelectricity. All of them are almost the same and based on classical theory of elasticity with the symmetrical stress tensor. Below the notation of a book [4] will be used. The basic equations can be represented in the conventional form.

The equations of motion:

$$\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{F} = \rho \ddot{\mathbf{u}}, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^T, \tag{1}$$

where $\boldsymbol{\tau}$ is the stress tensor, ρ is the mass density, $\mathbf{u}$ is the displacement vector.

The Poisson equations for crystal and vacuum respectively:

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{E}' = 0, \tag{2}$$

where $\mathbf{D}$ is the electrical induction, $\mathbf{E}'$ is the electrical field in the vacuum.

The piezoelectric effect equations

$$\boldsymbol{\tau} = \mathbf{C} \cdot \cdot \varepsilon - \mathbf{E} \cdot \mathbf{e}, \quad \mathbf{D} = \varepsilon \cdot\cdot \mathbf{e} + \boldsymbol{\epsilon} \cdot \mathbf{E}, \tag{3}$$

where $\mathbf{E}$ is the electrical field in the crystal, ε is the tensor of the linear deformation, $\mathbf{C}$ is the elasticity tensor, $\mathbf{e}$ is the tensor of piezoelectric modulus, $\boldsymbol{\epsilon}$ is the dielectric tensor.

This conventional theory is supposed to be able to give the description of all known experimental data. It is not so easy to confirm this point of view. In many practical cases we have the conspicuous discrepancy between the theoretical and experimental results — see, for example, the paper [5]. However the exact reasons of this discrepancy are not known. May be the reason is that the exact solutions of the system (1)–(3) can be found only in very particular cases. As a rule it is possible to find the approximation solution and because of this the solution does not coincide with an experiment.

However, it is quiet possible that the conventional theory (1)–(3) must be improved in some points. First of all, at least some piezoelectric crystals must be considered as dipole media. For example, it is easy calculate that the α-quartz is dipole crystal. It means that the rotational degrees of freedom must be taken into account.

The present paper is an attempt to consider piezoelectricity from this point of view.

3 The Euler Laws of Dynamics

Let us consider the elastic body. Let x_s be material (Lagrangian) coordinates. Let $\mathbf{r}(x_s)$ and $\mathbf{R}(x_s)$ be radius-vectors of the points in the reference and the actual configuration respectively. Bases in the reference and actual configurations are defined by following equations:

$$\mathbf{g}_s = \frac{\partial \mathbf{r}}{\partial x^s}, \quad \mathbf{G}_s(\mathbf{x}, t) = \frac{\partial \mathbf{R}(\mathbf{x}, t)}{\partial x^s}.$$

Let us introduce the reciprocal bases $\mathbf{g}^s$ $\mathbf{G}^s$ by means of the next expressions:

$$\mathbf{g}^s \cdot \mathbf{g}_m = \delta^s_m, \quad \mathbf{G}^s \cdot \mathbf{G}_m = \delta^s_m.$$

In the nonlinear theory it is necessary to use two the Hamilton operators [6]:

$$\overset{\circ}{\boldsymbol{\nabla}} = \mathbf{g}^s \frac{\partial}{\partial x^s}, \quad \boldsymbol{\nabla} = \mathbf{G}^s(\mathbf{x}, t) \frac{\partial}{\partial x^s}$$

for reference and actual configurations respectively.

The first law of dynamics by Euler in the integral form of momentum balance Law takes the following form:

$$\frac{d}{dt} \int_V \rho \dot{\mathbf{u}}_1 \, dV = \int_V \rho \mathbf{F} dV + \int_S \mathbf{T}_{(n)} \, dS, \tag{4}$$

where $\mathbf{u}(x_s) = \mathbf{R} - \mathbf{r}$ is displacement of the particle, $\mathbf{T}_{(n)}$ is the stress vector, $\mathbf{F}$ is an external force. Making use of Eq.(4) the Cauchy formulae

$$\mathbf{T}_{(n)} = \mathbf{n} \cdot \mathbf{T} \tag{5}$$

can be proved, where the second rank tensor $\mathbf{T}$ is called the Cauchy stress tensor. Taking into account Eqs. (4), (5) and the divergence theorem one can derive the local form of the first law of dynamics

$$\boldsymbol{\nabla} \cdot \mathbf{T} + \rho \mathbf{F} = \rho \ddot{\mathbf{u}}. \tag{6}$$

Here and below we suppose that the displacement vector $\mathbf{u}$ is small, i.e. we shall consider the linear theory. The stress tensor can be represented as decomposition

$$\mathbf{T} = \boldsymbol{\tau} - \frac{1}{2} \mathbf{q} \times \mathbf{I}, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^T, \quad \mathbf{q} = \mathbf{T}_\times \quad ((\mathbf{a} \otimes \mathbf{b})_\times \equiv \mathbf{a} \times \mathbf{b}), \tag{7}$$

where the vector $\mathbf{q}$ determines the antisymmetric part of the stress tensor, $\mathbf{I}$ is the unit tensor. In such a case the Eq.(6) can be rewritten in the form

$$\boldsymbol{\nabla} \cdot \boldsymbol{\tau} - \frac{1}{2} \boldsymbol{\nabla} \times \mathbf{q} + \rho \mathbf{F} = \rho \ddot{\mathbf{u}}. \tag{8}$$

In order to describe the rotations of a particle it is necessary to introduce the turn-tensor $\mathbf{P}$ or the vector of turn $\boldsymbol{\varphi}$. For the small rotations it is possible to use the next relation

$$\mathbf{P} \approx \mathbf{I} + \boldsymbol{\phi} \times \mathbf{I} \quad \Rightarrow \quad \boldsymbol{\omega} = \dot{\boldsymbol{\phi}},$$

where $\boldsymbol{\omega}$ is the angular velocity. According to the accepted model, particles of media are body-points and, thus, we must assign inertia tensor $\mathbf{J}$ for every such body-point. Let us note that the inertia tensor $\mathbf{J}$ is determined in the reference position, thus it has the constant value. In the actual position the inertia tensor must be calculated in the form $\mathbf{P} \cdot \mathbf{J} \cdot \mathbf{P}^T$. For small angular velocities we may use the following expression for kinetic momentum $\mathbf{K}_2$:

$$\mathbf{K}_2 = \rho(\mathbf{J} \cdot \dot{\boldsymbol{\phi}}(x, t) + \mathbf{r} \times \dot{\mathbf{u}}),$$

where $\boldsymbol{\phi}(x, t)$ is the vector of turn of the body-point.

The second law of dynamics can be written down in an integral form

$$\frac{d}{dt}\int_V \mathbf{K}_2 dV = \int_V \rho\left(\mathbf{r}\times\mathbf{F}+\mathbf{L}\right) dV + \int_S \left(\mathbf{r}\times\mathbf{T}_{(n)}+\boldsymbol{\mu}_{(n)}\right) dS, \tag{9}$$

where $\mathbf{L}$ is an external moment, $\boldsymbol{\mu}_{(n)}$ is the coupled stress vector. For the couple stress tensor $\boldsymbol{\mu}$ the Cauchy formula

$$\boldsymbol{\mu}_{(n)} = \mathbf{n}\cdot\boldsymbol{\mu} \tag{10}$$

is valid. After some standard transformations one can obtain the kinetic moment balance equation in local form

$$\boldsymbol{\nabla}\cdot\boldsymbol{\mu}+\mathbf{q}+\rho\mathbf{L} = \rho\mathbf{J}\cdot\ddot{\boldsymbol{\varphi}}. \tag{11}$$

The equations (8) and (11) are well known in the theory of micropolar media. However, we shall use a special form of the coupled stress tensor

$$\boldsymbol{\mu} = \mathbf{m}\times\mathbf{I}. \tag{12}$$

This means that the coupled stress tensor is antisymmetric. The vector $\mathbf{m}$ determines the antisymmetric part of tensor $\boldsymbol{\mu}$. Of course, the assumption (12) must be justified. Maybe it will be necessary to do this justification in the future. Substituting representation (12) into equation (11) we obtain

$$\boldsymbol{\nabla}\times\mathbf{m}+\mathbf{q}+\rho\mathbf{L} = \rho\mathbf{J}\cdot\ddot{\boldsymbol{\varphi}}. \tag{13}$$

4 The equation of the Energy Balance

Now we have to discuss the energy balance equation. The integral form of this equation can be represented as

$$\frac{d}{dt}\int_V \{\frac{1}{2}\rho\dot{\mathbf{u}}^2+\frac{1}{2}\rho\dot{\boldsymbol{\varphi}}\cdot\mathbf{J}\cdot\dot{\boldsymbol{\varphi}}+\rho\mathbb{U}\}dV = \int_V \{\rho\mathbf{F}\cdot\dot{\mathbf{u}}+\rho\mathbf{L}\cdot\dot{\boldsymbol{\varphi}}+Q\}dV + \\ + \int_S \{\mathbf{T}_{(n)}\cdot\dot{\mathbf{u}}+\boldsymbol{\mu}_{(n)}\cdot\dot{\boldsymbol{\varphi}}+\mathbf{H}\cdot\mathbf{n}\}dS, \tag{14}$$

where Q is the volume external energy supply and $\mathbf{H}$ is energy flow vector.

Equation (14) may be transformed to the following form

$$\int_V \{\rho\dot{\mathbb{U}}+\dot{\mathbf{u}}\cdot(\rho\ddot{\mathbf{u}}-\rho\mathbf{F}-\boldsymbol{\nabla}\cdot\mathbf{T})+\dot{\boldsymbol{\varphi}}\cdot(\rho\mathbf{J}\cdot\ddot{\boldsymbol{\varphi}}-\rho\mathbf{L}-\boldsymbol{\nabla}\cdot\boldsymbol{\mu})- \\ -\mathbf{T}^T\cdot\cdot\boldsymbol{\nabla}\dot{\mathbf{u}}-\boldsymbol{\mu}^T\cdot\cdot\boldsymbol{\nabla}\dot{\boldsymbol{\varphi}}-Q-\boldsymbol{\nabla}\cdot\mathbf{H}\}dV = 0. \tag{15}$$

Making use the laws of dynamics the equation (15) may be rewritten in the local form

$$\rho\dot{\mathbb{U}} = \mathbf{T}^T\cdot\cdot\boldsymbol{\nabla}\dot{\mathbf{u}}-\mathbf{q}\cdot\dot{\boldsymbol{\varphi}}+\boldsymbol{\mu}^T\cdot\cdot\boldsymbol{\nabla}\dot{\boldsymbol{\varphi}}+\boldsymbol{\nabla}\cdot\mathbf{H}+Q. \tag{16}$$

It is easy to proof the identity

$$\mathbf{T}^T\cdot\cdot\boldsymbol{\nabla}\dot{\mathbf{u}}-\mathbf{q}\cdot\dot{\boldsymbol{\varphi}} = \boldsymbol{\tau}\cdot\cdot\dot{\boldsymbol{\varepsilon}}-\mathbf{q}\cdot\dot{\boldsymbol{\theta}},$$

where

$$\boldsymbol{\varepsilon} \equiv \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathsf{T}}\right), \quad \boldsymbol{\theta} \equiv \boldsymbol{\phi} - \frac{1}{2}\nabla \times \mathbf{u}. \tag{17}$$

The symmetric tensor $\boldsymbol{\varepsilon}$ is called the tensor of linear deformation and the vector $\boldsymbol{\theta}$ is the turn of the body-point with respect to its small neighborhood. Equation (16) takes the form

$$\rho\dot{\mathbb{U}} = \boldsymbol{\tau} \cdot\cdot\, \dot{\boldsymbol{\varepsilon}} + \mathbf{q} \cdot \dot{\boldsymbol{\gamma}} + \boldsymbol{\mu}^{\mathsf{T}} \cdot\cdot\, \nabla \dot{\boldsymbol{\phi}} + \nabla \cdot \mathbf{H} + \mathrm{Q}. \tag{18}$$

Making use an idea of the paper [8], let us introduce into consideration two vectors $\mathbf{E}$ and $\mathbf{D}$ such that

$$\nabla \cdot \mathbf{H} + \mathrm{Q} = \mathbf{E} \cdot \dot{\mathbf{D}}, \tag{19}$$

where the vectors $\mathbf{E}$ and $\mathbf{D}$ will be called the electric field vector and the electric displacement vector respectively. Thus, we may write down the energy balance equation in the final form

$$\rho\dot{\mathbb{U}} = \boldsymbol{\tau} \cdot\cdot\, \dot{\boldsymbol{\varepsilon}} - \mathbf{q} \cdot \dot{\boldsymbol{\theta}} + \boldsymbol{\mu}^{\mathsf{T}} \cdot\cdot\, \nabla \dot{\boldsymbol{\phi}} + \mathbf{E} \cdot \dot{\mathbf{D}}. \tag{20}$$

From this it follows that the volume density of intrinsic energy $\rho\mathbb{U}$ depends on the arguments: $\boldsymbol{\varepsilon}$, $\boldsymbol{\theta}$, $\mathbf{D}$ and $\nabla\boldsymbol{\phi}$. In many cases it is more convenient to consider the vector $\mathbf{E}$ as independent variable instead of $\mathbf{D}$. In such a case it would be better to use the free energy

$$\rho\mathbb{F} = \rho\mathbb{U} - \mathbf{E} \cdot \mathbf{D}. \tag{21}$$

Free energy depends on $\boldsymbol{\varepsilon}$, $\boldsymbol{\theta}$, $\mathbf{E}$ and $\nabla\boldsymbol{\phi}$. In terms of the free energy equation (20) may be rewritten as

$$\rho\dot{\mathbb{F}} = \boldsymbol{\tau} \cdot\cdot\, \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\mu}^{\mathsf{T}} \cdot\cdot\, \nabla \dot{\boldsymbol{\phi}} - \mathbf{q} \cdot \dot{\boldsymbol{\theta}} - \mathbf{D} \cdot \dot{\mathbf{E}}. \tag{22}$$

We have

$$\rho\dot{\mathbb{F}} = \left(\frac{\partial \rho\mathbb{F}}{\partial \boldsymbol{\varepsilon}}\right)^{\mathsf{T}} \cdot\cdot\, \dot{\boldsymbol{\varepsilon}} + \frac{\partial \rho\mathbb{F}}{\partial \boldsymbol{\theta}} \cdot \dot{\boldsymbol{\theta}} + \frac{\partial \rho\mathbb{F}}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} + \left(\frac{\partial \rho\mathbb{F}}{\partial \nabla\boldsymbol{\phi}}\right)^{\mathsf{T}} \cdot \nabla\dot{\boldsymbol{\phi}}. \tag{23}$$

From the comparison of equations (22) and (23) the Cauchy – Green relations follow

$$\boldsymbol{\tau} = \frac{\partial \rho\mathbb{F}}{\partial \boldsymbol{\varepsilon}}, \quad \boldsymbol{\mu} = \frac{\partial \rho\mathbb{F}}{\partial \nabla\boldsymbol{\phi}}, \quad \mathbf{q} = -\frac{\partial \rho\mathbb{F}}{\partial \boldsymbol{\theta}}, \quad \mathbf{D} = -\frac{\partial \rho\mathbb{F}}{\partial \mathbf{E}}. \tag{24}$$

Below the natural state hypothesis is accepted. This means absence of stress when strain is equal to zero. In such a case we have the representation for the free energy in the form

$$\begin{aligned}\rho\mathbb{F} = \rho\mathbb{F}_0 + \frac{1}{2}\boldsymbol{\varepsilon} \cdot\cdot\, \mathbf{C} \cdot\cdot\, \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \cdot\cdot\, \mathbf{M} \cdot \boldsymbol{\theta} + \frac{1}{2}\boldsymbol{\theta} \cdot \mathbf{P} \cdot \boldsymbol{\theta} + \boldsymbol{\varepsilon} \cdot\cdot\, \mathbf{N} \cdot \mathbf{E} + \frac{1}{2}\mathbf{E} \cdot \boldsymbol{\epsilon} \cdot \mathbf{E} + \boldsymbol{\theta} \cdot \mathbf{X} \cdot \mathbf{E} + \\ + \frac{1}{2}\nabla\boldsymbol{\phi} \cdot\cdot\, \boldsymbol{\Phi}^{\mu} \cdot\cdot\, \nabla\boldsymbol{\phi} + \boldsymbol{\theta} \cdot \boldsymbol{\Phi}^{\times} \cdot\cdot\, \nabla\boldsymbol{\phi} + \boldsymbol{\epsilon} \cdot\cdot\, \boldsymbol{\Phi}^{\tau} \cdot\cdot\, \nabla\boldsymbol{\phi} + \mathbf{E} \cdot \boldsymbol{\Phi}^{\mathrm{E}} \cdot\cdot\, \nabla\boldsymbol{\phi}.\end{aligned} \tag{25}$$

The intrinsic energy must be a positively defined function. This means that the known restrictions must be superposed on the tensors of elasticity: $\mathbf{C}$, $\mathbf{M}$, $\mathbf{N}$, $\mathbf{X}, \mathbf{P}$, $\boldsymbol{\Phi}^{\mu}$, $\boldsymbol{\Phi}^{\times}$, $\boldsymbol{\Phi}^{\mathrm{E}}$, $\boldsymbol{\Phi}^{\tau}$.

Substituting expression (25) into the Cauchy – Green relations (24) we shall get the stress – strain relations

$$\boldsymbol{\tau} = \frac{\partial \rho \mathbb{F}}{\partial \boldsymbol{\varepsilon}} = \mathbf{C} \cdot \cdot \boldsymbol{\varepsilon} + \mathbf{M} \cdot \boldsymbol{\theta} + \mathbf{N} \cdot \mathbf{E} + \boldsymbol{\Phi}^{\tau} \cdot \cdot \nabla \boldsymbol{\phi}, \quad (26)$$

$$\mathbf{q} = -\frac{\partial \rho \mathbb{F}}{\partial \boldsymbol{\theta}} = -\boldsymbol{\varepsilon} \cdot \cdot \mathbf{M} - \mathbf{P} \cdot \boldsymbol{\theta} - \mathbf{X} \cdot \mathbf{E} - \boldsymbol{\Phi}^{\times} \cdot \cdot \nabla \boldsymbol{\phi}, \quad (27)$$

$$\mathbf{D} = -\frac{\partial \rho \mathbb{F}}{\partial \mathbf{E}} = -\boldsymbol{\varepsilon} \cdot \cdot \mathbf{N} - \boldsymbol{\theta} \cdot \mathbf{X} - \boldsymbol{\epsilon} \cdot \mathbf{E} - \boldsymbol{\Phi}^{E} \cdot \nabla \boldsymbol{\phi}, \quad (28)$$

$$\boldsymbol{\mu} = \frac{\partial \rho \mathbb{F}}{\partial \nabla \boldsymbol{\phi}} = \boldsymbol{\Phi}^{\mu} \cdot \cdot \nabla \boldsymbol{\phi} + \boldsymbol{\theta} \cdot \boldsymbol{\Phi}^{\times} + \mathbf{E} \cdot \boldsymbol{\Phi}^{E} + \boldsymbol{\epsilon} \cdot \cdot \boldsymbol{\Phi}^{\tau}. \quad (29)$$

Tensors $\mathbf{C}$, $\mathbf{M}$, $\mathbf{N}$, $\mathbf{X}$, $\mathbf{P}$, $\boldsymbol{\Phi}^{\mu}$, $\boldsymbol{\Phi}^{\times}$, $\boldsymbol{\Phi}^{E}$ and $\boldsymbol{\Phi}^{\tau}$ describe the physical properties of the material under consideration.

5 The special form of the energy balance equation

In order to simplify the theory let us accept the assumption (12). In such a case, instead of equation (22) we obtain

$$\rho \dot{\mathbb{F}} = \boldsymbol{\tau} \cdot \cdot \dot{\boldsymbol{\varepsilon}} - \mathbf{m} \cdot \dot{\boldsymbol{\gamma}} - \mathbf{q} \cdot \dot{\boldsymbol{\theta}} - \mathbf{D} \cdot \dot{\mathbf{E}}, \quad (30)$$

where

$$\boldsymbol{\gamma} \equiv \nabla \times \boldsymbol{\phi}. \quad (31)$$

The Cauchy – Green relations (24) take a form

$$\boldsymbol{\tau} = \frac{\partial \rho \mathbb{F}}{\partial \boldsymbol{\varepsilon}}, \quad \mathbf{m} = -\frac{\partial \rho \mathbb{F}}{\partial \boldsymbol{\gamma}}, \quad \mathbf{q} = -\frac{\partial \rho \mathbb{F}}{\partial \boldsymbol{\theta}}, \quad \mathbf{D} = -\frac{\partial \rho \mathbb{F}}{\partial \mathbf{E}}. \quad (32)$$

The free energy (25) will be accepted in the more simple form

$$\rho \mathbb{F} = \rho \mathbb{F}_0 + \frac{1}{2} \boldsymbol{\varepsilon} \cdot \cdot \mathbf{C} \cdot \cdot \boldsymbol{\varepsilon} + \frac{1}{2} \boldsymbol{\theta} \cdot \mathbf{P} \cdot \boldsymbol{\theta} + \frac{1}{2} \chi \boldsymbol{\gamma} \cdot \boldsymbol{\gamma} + \frac{1}{2} \mathbf{E} \cdot \boldsymbol{\epsilon} \cdot \mathbf{E} + \\ + \boldsymbol{\varepsilon} \cdot \cdot \mathbf{M} \cdot \boldsymbol{\theta} + \boldsymbol{\varepsilon} \cdot \cdot \mathbf{N} \cdot \mathbf{E} + \boldsymbol{\theta} \cdot \mathbf{X} \cdot \mathbf{E}, \quad (33)$$

where χ is the physical constant. The stress – strain relations (26)–(29) takes a form

$$\boldsymbol{\tau} = \frac{\partial \rho \mathbb{F}}{\partial \boldsymbol{\varepsilon}} = \mathbf{C} \cdot \cdot \boldsymbol{\varepsilon} + \mathbf{M} \cdot \boldsymbol{\theta} + \mathbf{N} \cdot \mathbf{E}, \quad (34)$$

$$\mathbf{q} = -\frac{\partial \rho \mathbb{F}}{\partial \boldsymbol{\theta}} = -\boldsymbol{\varepsilon} \cdot \cdot \mathbf{M} - \mathbf{P} \cdot \boldsymbol{\theta} - \mathbf{X} \cdot \mathbf{E}, \quad (35)$$

$$\mathbf{D} = -\frac{\partial \rho \mathbb{F}}{\partial \mathbf{E}} = -\boldsymbol{\varepsilon} \cdot \cdot \mathbf{N} - \boldsymbol{\theta} \cdot \mathbf{X} - \boldsymbol{\epsilon} \cdot \mathbf{E}, \quad (36)$$

$$\mathbf{m} = -\frac{\partial \rho \mathbb{F}}{\partial \boldsymbol{\gamma}} = -\chi \boldsymbol{\gamma}. \quad (37)$$

Now we have to find the general form of the tensors $\mathbf{C}$, $\mathbf{P}$, $\boldsymbol{\epsilon}$, $\mathbf{M}$, $\mathbf{N}$, $\mathbf{X}$. For this it is necessary to use the symmetry requirements.

6 Symmetry and the tensor transformations

When using the symmetry groups we have to take into account the type of a tensor. There exist tensors of two different types: polar and axial tensors. Axial tensor depends on the choice of the orientation in 3D space, but the polar tensor does not depend on the choice of the orientation in 3D space. Usually the axial vector associates with rotations and the polar vector associates with translations in the space. As a matter of fact we do not know the type of the electrical field vector. In electrodynamics the type of the vector **E** does not matter [10]. In electro-elasticity the type of the vector **E** is very important, since the type of the tensors **N**, **X** depends on the type of the vector **E**. Let there be given tensors **A** and **B**, the symmetry groups of which are the same, but the types of these tensors are different. In such a case the structures of tensors **A** and **B** will be different. If the vector **E** is polar one, then in the case under consideration the tensors **C**, **P**, $\boldsymbol{\epsilon}$, **N**, **J** are the polar (euclidian) tensors and **M**, **X** are the axial tensors. If the vector **E** is axial one, then the tensors **C**, **P**, $\boldsymbol{\epsilon}$, **X**, **J** are the polar (euclidian) tensors and **M**, **N** are the axial tensors. This fact can be established by means of experiment.

Let us accept the definition [9]

Definition 1. *Orthogonal transformation of a tensor* **S** *of a rank* k *is a tensor*

$$\mathbf{S}' \equiv (\det \mathbf{Q})^{\alpha} \otimes_1^k \mathbf{Q} \cdot \mathbf{S} \equiv (\det \mathbf{Q})^{\alpha} S^{i_1 \ldots i_k} \mathbf{Q} \cdot \mathbf{g}_{i_1} \otimes \ldots \otimes \mathbf{Q} \cdot \mathbf{g}_{i_k}, \tag{38}$$

where $\alpha = 0$, if the tensor **S** is a polar tensor and $\alpha = 1$, if the tensor **S** is an axial tensor. Let us accept the notation for a turn-tensor

$$\mathbf{Q}(\alpha \mathbf{n}) = (1 - \cos\alpha)\, \mathbf{n} \otimes \mathbf{n} + \cos\alpha\, \mathbf{I} + \sin\alpha\, \mathbf{n} \times \mathbf{I},$$

where α is the angle of turn and the unit vector **n** determines the axis of turn. The tensors

$$\mathbf{Q} = -\mathbf{I}, \quad \mathbf{Q} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$$

are called the central inversion tensor and the mirror inversion tensor respectively.

Let us define [8] the symmetry group of a tensor **S**.

Definition 2. *Symmetry group of a tensor* **S** *is the set of the orthogonal tensors,* $\mathbf{Q}_s$*, which are the orthogonal solutions of equation*

$$(\det \mathbf{Q})^{\alpha} \otimes_1^k \mathbf{Q} \cdot \mathbf{S} = \mathbf{S}, \tag{39}$$

where **S** *is the given tensor.*

If a tensor **S** is known, then it is easy to find its group of symmetry. If we know the symmetry group of some tensor, then it is possible to construct a general form of the tensor with this symmetry group. To this end we must use the Curie – Neumann principle.

Curie – Neumann principle: *The symmetry group of the cause is a sub-set of the symmetry group of the consequence.*

In our case we are working with a certain piezoelectric crystal, for example, α-quartz. According to the Curie – Neumann Principle, the symmetry group of tensors **C**, **M** **P** may be equivalent or wider than the symmetry group of the crystal. Additional symmetry elements may appear as effects of shape, etc. Since we consider infinite 3D crystal, it is possible to find out the form of tensors using the invariance about all symmetry elements inherited by crystal structure. Numerical values of components must be found out experimentally.

For example let us consider 3-rank tensor **M** and present it in the following form

$$\mathbf{M} = M^{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k.$$

Transformed tensor:

$$\mathbf{M}' = (\det\mathbf{Q})^{\alpha} M^{ijk}\mathbf{Q}\cdot\mathbf{e}_i \otimes \mathbf{Q}\cdot\mathbf{e}_j \otimes \mathbf{Q}\cdot\mathbf{e}_k.$$

If **Q** is the symmetry element of the crystal it is necessary to require $\mathbf{M}' = \mathbf{M}$. In the other form:

$$M^{ijk}\left[(\det\mathbf{Q})^{\alpha}\mathbf{Q}\cdot\mathbf{e}_i \otimes \mathbf{Q}\cdot\mathbf{e}_j \otimes \mathbf{Q}\cdot\mathbf{e}_k \otimes -\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k\right] = 0 \tag{40}$$

We have 3^3 equations for every symmetry element, but the most of them are identities. If the crystal under consideration has n symmetry elements, the number of equations in (40) should be $n3^3$.

Now let us apply the conditions (40) to the tensor **N**. Let some crystal has the inversion tensor $(-\mathbf{I})$ as its element of symmetry. Let the vector **E** is the polar vector. In such a case the tensor **N** must be polar as well. The conditions (40) gives $\mathbf{N} = \mathbf{0}$. It means that the piezoeffect for this kind of crystal is impossible. If the vector **E** is the axial vector, then the tensor **N** must be axial as well. The conditions (40) are identities. It means that the piezoeffect for this kind of crystal is possible. This is an experimental way to establish the type of the vector **E**. If we find out the piezoelectric material with the central symmetry, then the vector **E** must be axial. We do not if there exist the piezoelectric material of such a kind, but theoretically such piezoelectric material may exist. From the other hand, it is well known that there exist the piezoelectric materials with two planes of the mirror symmetry. Let the tensors

$$\mathbf{Q}_1 = \mathbf{I} - \mathbf{e}_1 \otimes \mathbf{e}_1, \quad \mathbf{Q}_2 = \mathbf{I} - \mathbf{e}_2 \otimes \mathbf{e}_2 \tag{41}$$

be the symmetry elements of some crystal. According to the Curie – Neumann Principle these tensors must belong to the symmetry group of the tensor **N**. If tensor **N** is a polar tensor, then we have

$$\mathbf{N} = \left(N^{113}\mathbf{e}_1 \otimes \mathbf{e}_1 + N^{223}\mathbf{e}_2 \otimes \mathbf{e}_2 + N^{333}\mathbf{e}_3 \otimes \mathbf{e}_3\right) \otimes \mathbf{e}_3 + \\ + N^{131}\left(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1\right) \otimes \mathbf{e}_1 + N^{232}\left(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2\right) \otimes \mathbf{e}_2. \tag{42}$$

If tensor **N** is an axial tensor, then we have another representation

$$\mathbf{N} = N^{231}\left(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2\right) \otimes \mathbf{e}_1 + \\ + N^{132}\left(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1\right) \otimes \mathbf{e}_2 + N^{123}\left(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1\right) \otimes \mathbf{e}_3. \tag{43}$$

In conventional theory of the piezoelectricity expression (42) is used and the vector **E** is supposed to be a polar vector. However, there is some reasons in order to consider the vector **E** to be an axial one. Thus it is possible that expression (43) will be better to describe the real crystals. In any case the situation must be studied more carefully.

If the symmetry group of a crystal contains only turns, then the type of the vector **E** does not matter. Below we derive the results of treating the system (40) for α-quartz which belongs to class 32. There are two symmetry elements of class 32 structure: turn around axis x_3 about angle $2\pi/3$ and turn around axis x_1 about angle π.

Any two-rank tensor of quartz must have the following form:

$$\mathbf{T}^{(2)} = t_1(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + t_2\mathbf{e}_3 \otimes \mathbf{e}_3 = t_1\mathbf{I} + (t_2 - t_1)\mathbf{e}_3 \otimes \mathbf{e}_3. \qquad (44)$$

Also, any three-rank tensor of quartz must have the form:

$$\mathbf{T}^{(3)} = t_0(\mathbf{e}_1 \otimes \mathbf{a} - \mathbf{e}_2 \otimes \mathbf{b}) + t_1\mathbf{e}_3 \otimes \mathbf{c} + t_2\mathbf{c} \otimes \mathbf{e}_3 + t_3(\mathbf{e}_1 \otimes \mathbf{e}_3 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_3 \otimes \mathbf{e}_1), \qquad (45)$$

where

$$\mathbf{a} = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2, \quad \mathbf{b} = \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1, \quad \mathbf{c} = \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1.$$

The four-rank tensor of quartz have rather complicated form and, thus we just mention that it has 14 independent components. However when we consider the tensor of elasticity the last one has simple form due to symmetry of the stress tensor and the strain tensor. In this case we have only 6 independent components. It is common in symmetric elasticity theory to perform the 4-rank elasticity tensor as 6×6 matrix.

7 The simplest piezoelectric media

There are a lot of crystals which have the piezoelectric effect. Piezoelectricity is the result of interaction of crystal with electromagnetic field. In this work a piezoelectric crystal is supposed to be dipole crystal. This means that low-level cell has dipole properties. Electric field influences over dipole and creates a torque. There are two ways to impart energy to the crystal: either through an external force, or through an external torque. In order to complete the formulation of the theory we have to calculate the volume density of force $\rho\mathbf{F}$ in equation (8) and the volume density of torque $\rho\mathbf{L}$ in equations (11) or (13). To this end we can calculate the power of the external forces by two ways

$$\rho\left(\mathbf{F} \cdot \dot{\mathbf{u}} + \mathbf{L} \cdot \dot{\boldsymbol{\phi}}\right) = q_+\mathbf{D}_+ \cdot \mathbf{v}_+ + q_-\mathbf{D}_- \cdot \mathbf{v}_-, \qquad (46)$$

where the Lorentz forces are taken into account. Each point of the media is neutral dipole. Dipole is placed along the vector $\mathbf{d}_0$. Then we have

$$q_+ = -q_- = q, \quad \mathbf{D}_+ = \mathbf{D}_- = \mathbf{D}, \quad \mathbf{v}_+ = \dot{\mathbf{u}} + \frac{1}{2}\dot{\boldsymbol{\phi}} \times \mathbf{d}_0, \quad \mathbf{v}_- = \dot{\mathbf{u}} - \frac{1}{2}\dot{\boldsymbol{\phi}} \times \mathbf{d}_0.$$

Substituting these expressions into equation (46) we obtain

$$\rho\left(\mathbf{F} \cdot \dot{\mathbf{u}} + \mathbf{L} \cdot \dot{\boldsymbol{\phi}}\right) = q\left(\mathbf{d_0} \times \mathbf{D}\right) \cdot \dot{\boldsymbol{\phi}} \quad \Rightarrow \quad \rho\mathbf{F} = \mathbf{0}, \quad \rho\mathbf{L} = \mathbf{d} \times \mathbf{D}, \qquad (47)$$

where $\mathbf{d} = q\mathbf{d}_0$ is the physical characteristic of the material under consideration.

Let us write down the complete system of the piezoelectric equations.

Equations of motion

$$\nabla \cdot \boldsymbol{\tau} - \frac{1}{2}\nabla \times \mathbf{q} + \rho\mathbf{F} = \rho\ddot{\mathbf{u}}, \quad \nabla \times \mathbf{m} + \mathbf{q} + \mathbf{d} \times \mathbf{D} = \rho\,\mathbf{J} \cdot \ddot{\boldsymbol{\phi}}, \quad \nabla \cdot \mathbf{D} = \mathbf{0}. \qquad (48)$$

The Cauchy – Green relations

$$\boldsymbol{\tau} = \frac{\partial\rho\mathbb{F}}{\partial\varepsilon}, \quad \mathbf{m} = -\frac{\partial\rho\mathbb{F}}{\partial\gamma}, \quad \mathbf{q} = -\frac{\partial\rho\mathbb{F}}{\partial\theta}, \quad \mathbf{D} = -\frac{\partial\rho\mathbb{F}}{\partial\mathbf{E}}. \qquad (49)$$

In the given paper we are not going to discuss the theory of piezoelectricity for the real crystals of a general form. Our aim is only to discuss the main features of a new theory. By this reason let us consider the simplest expression for the free energy

$$\rho\mathbb{F} = \mu\varepsilon\cdot\cdot\varepsilon + \frac{1}{2}\lambda\,(\mathrm{tr}\varepsilon)^2 + \frac{1}{2}p\,\theta\cdot\theta + \frac{1}{2}\chi\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} + \frac{1}{2}\epsilon\mathbf{E}\cdot\mathbf{E} + \varepsilon\cdot\cdot\mathbf{N}\cdot\mathbf{E} + \theta\cdot\mathbf{X}\cdot\mathbf{E}. \quad (50)$$

In this representation only the terms relating with the piezoeffect were taken into account in general form.

The stress – strain relations

$$\boldsymbol{\tau} = 2\mu\,\varepsilon + \lambda\,\mathrm{tr}\varepsilon\mathbf{I} + \mathbf{N}\cdot\mathbf{E}, \quad \mathbf{q} = -p\,\theta - \mathbf{X}\cdot\mathbf{E}, \quad \mathbf{m} = -\chi\,\boldsymbol{\gamma}, \quad \mathbf{D} = -\epsilon\mathbf{E} - \varepsilon\cdot\cdot\mathbf{N}. \quad (51)$$

The geometrical equations

$$\varepsilon = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \quad \theta = \boldsymbol{\phi} - \frac{1}{2}\nabla\times\mathbf{u}, \quad \boldsymbol{\gamma} = \nabla\times\boldsymbol{\phi}, \quad \mathbf{E} = \nabla\varphi, \quad (52)$$

where φ is an electrostatics potential.

The classical theory of the piezoelectricity follows from equations (48)–(52) under the next conditions

$$\boldsymbol{\phi} = \frac{1}{2}\nabla\times\mathbf{u}, \quad \mathbf{X} = \mathbf{0}, \quad \chi = 0, \quad \mathbf{d}\times\mathbf{D} = \mathbf{0}, \quad \mathbf{J} = \mathbf{0}. \quad (53)$$

While equations (48)–(52) can not be applied to the crystals of general form, nevertheless they contain a several different versions of the piezoelectricity theory. At the moment it is impossible to accept the final decision what theory is better. First of all, we do not know the type of the electrical field vector **E**. However this is very important for the piezoelectricity theory. As a matter of fact it is necessary to construct electrodynamics on the base of rational mechanics. P. Zhilin is quit sure that there exist only one possibility: the vector **E** is an axial vector (this means that a charge is a pseudoscalar). This fact follows from an unpublished yet work by Zhilin on electrodynamics. In any way we have to consider the two possibilities: both when the vector **E** is a polar vector and when the vector **E** is an axial vector. Besides, from equations (48)–(52) it follows that the piezoeffect penetrate into the theory by means of two way: either when

$$\mathbf{N} \neq \mathbf{0}, \quad \mathbf{X} = \mathbf{0} \quad (54)$$

or when

$$\mathbf{N} = \mathbf{0}, \quad \mathbf{X} \neq \mathbf{0}. \quad (55)$$

Of course both tensor **N** and tensor **X** may be in general different from zero.

Let us consider the particular case. Let the dipole direction be parallel to the optic axis x_3: $\mathbf{d} = d\mathbf{e}_3$. Let the symmetry group of the piezoelectric properties of a crystal contains the tensors (41) and any turn around the axis $\mathbf{e}_3$.

If the vector **E** is a polar vector, then the tensor **N** is a polar tensor, but the tensor **X** is an axial one. In such a case we have

$$\mathbf{N} = [N_1\mathbf{I} + (N_2 - N_1)\mathbf{e}_3\otimes\mathbf{e}_3]\otimes\mathbf{e}_3 + N_3\,[\mathbf{e}_1\otimes\mathbf{e}_3\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_3\otimes\mathbf{e}_2 + \\ + \mathbf{e}_3\otimes(\mathbf{I} - \mathbf{e}_3\otimes\mathbf{e}_3)], \qquad \mathbf{X} = X_1\,\mathbf{e}_3\times\mathbf{I}, \quad (56)$$

where N_1, N_2, N_3, X_1 are true scalars.

If the vector **E** is an axial vector, then the tensor **N** is an axial tensor, but the tensor **X** is a polar one. In such a case we have

$$\mathbf{N} = N_1\,[\mathbf{e}_2 \otimes \mathbf{e}_3 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_3 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 \times \mathbf{I}]\,, \quad \mathbf{X} = X_1\,\mathbf{I} + (X_2 - X_1)\mathbf{e}_3 \otimes \mathbf{e}_3, \tag{57}$$

where N_1, X_1, X_2 are true scalars.

From the comparison of the expressions (56) and (57) we see the significant difference between them. It is important that this difference can be established by means of experiment. This means that it is possible to find out the type of the electric field vector **E**. In order to use the experimental data we have to solve some concrete problems and to determine what kind of theory is better to describe the experimental data. This way leads to rather complicated dispersion equations which may be represented as the roots of the equation of an order 6. It has 6 different roots, which gives 6 dispersion curves. There are 3 acoustic and 3 optic curves. Classical theory gives us only acoustic curves.

8 Conclusion

In the paper two different versions of the piezoelectricity theory were derived. Both of them are new. Now we have to investigate the consequences from these theories and compare them with the experimental data. From theoretical point of view the most interesting result is to clear if the electric field **E** is a polar vector or it is an axial vector. At the moment authors are not ready to formulate the final results, since they must be verified very carefully.

As an illustration let us consider the simplest cases, when the tensor **N** is equal to zero. We have to consider two cases.

The first case: the electric field vector **E** is a polar vector. In such a case the strain – strain relations (51) take a form

$$\boldsymbol{\tau} = 2\mu\,\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}\boldsymbol{\varepsilon}\mathbf{I}, \quad \mathbf{q} = -p\,\boldsymbol{\theta} - X_1\mathbf{e}_3 \times \mathbf{E}, \quad \mathbf{m} = -\chi\,\boldsymbol{\gamma}, \quad \mathbf{D} = -\epsilon\mathbf{E}. \tag{58}$$

The second case: the electric field vector **E** is an axial vector. In this case the strain – strain relations (51) take the next form

$$\boldsymbol{\tau} = 2\mu\,\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}\boldsymbol{\varepsilon}\mathbf{I},\ \mathbf{q} = -p\,\boldsymbol{\theta} - X_1\mathbf{E} - (X_2 - X_1)(\mathbf{E}\cdot\mathbf{e}_3)\mathbf{e}_3,\ \mathbf{m} = -\chi\,\boldsymbol{\gamma},\ \mathbf{D} = -\epsilon\mathbf{E}. \tag{59}$$

If the direction of vector **E** coincides with the direction of the vector $\mathbf{e}_3$, then the piezoeffect is absent in the case (58). However in the case (59) the effect will be present. If we shall be able to find out a crystal with the property (59), then it will be established that the electric field vector **E** is the axial vector, what is very important from the theoretical point of view. It is obvious that the dispersion curves will be quiet different for the cases (58) and (59). We do not actually know if there exist the piezoelectric crystals with such properties. But the existence of such crystals is theoretically possible. Let us point out that the cases under consideration differ from the classical case (1)–(3) very significantly. Up to present time only classical theory was verified by means of the experimental data. We may hope that a new theory — not necessary like the cases (58) and (59) — will be able to describe the experimental data better and simpler then the classical theory. In any case this possibility must be investigated in all details.

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A Micro-Polar Theory for Binary Media with Application to Flow of Fiber Suspensions*

Abstract

A phase-transitional flow takes place during the filling stage by injection molding of short-fiber reinforced thermoplastics. The mechanical properties of the final product are highly dependent on the flow-induced distribution and orientation of particles. Therefore, modelling of the flow which allows to predict the formation of fiber microstructure is of particular importance for analysis and design of load bearing components.

The aim of this paper is a discussion of existing models which characterize the behavior of fiber suspensions as well as the derivation of a model which treats the filling process as a phase-transitional flow of a binary medium consisting of fluid particles (liquid constituent) and immersed particles-fibers (solid-liquid constituent). The particle density and the mass density are considered as independent functions in order to account for the phenomenon of sticking of fluid particles to fibers. The liquid constituent is treated as a non-polar viscous fluid, but with a non-symmetric stress tensor. The state of the solid-liquid constituent is described by the antisymmetric stress tensor and the antisymmetric moment stress tensor. The forces of viscous friction between the constituents are taken into account. The equations of motion are formulated for open physical system in order to consider the phenomenon of sticking. The chemical potential is introduced based on the reduced energy balance equation. The second law of thermodynamics is formulated by means of two inequalities under the assumption that the constituents may have different temperatures. In order to take into account the phase transitions of the liquid-solid type which take place during the flow process a model of compressible fluid and a constitutive equation for the pressure are proposed. Finally, the set of governing equations which should be solved numerically in order to simulate the filling process are summarized. The special cases of these equations are discussed by introduction of restricting assumptions.

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1 Introduction

1.1 Motivation

The use of short fiber reinforced thermoplastics (fiber length around 0.1–1 mm, fiber diameter around 0.01 mm, fiber volume fraction 15–40%) has been rapidly increasing during the last years in many industrial branches, e.g. automobile industry, pump industry, etc. [22, 30, 37, 41]. Various load bearing components (usually thin-walled structures) are manufactured from these materials by injection molding. This manufacturing process is of particular interest because of highly automated production, relatively short cycle time and low production costs. Furthermore, the principal advantage of this process over other methods of composites manufacturing is the possibility of mass production of articles with a desired geometrical complexity. However, the mechanical properties of particle reinforced materials are quite poor if compared with those of materials reinforced by continuous fibers. In addition, the stiffness and the strength of short-fiber reinforced composites and thin-walled structures manufactured from these materials are highly dependent on the orientation and the distribution of particles. As show many experimental observations, the fiber orientation microstructure, induced by injection molding has significant spatial variations within the part, e.g. [8, 36, 41]. The orientation of fibers and the distribution of fibers density depend on many factors including the material type, process conditions and the geometry of the mold cavity, e.g. [24]. Therefore, the key step in the preliminary design of load bearing components lies in the prediction of the fiber orientation pattern for given manufacturing conditions.

Figure 1 illustrates schematically the basic units of a typical injection molding machine and the main stages of the processing cycle. During the filling stage, Fig. 1a, the rotating screw moves forward and pushes the melt into the mold cavity. After the complete filling of the cavity, a pressure is exerted and hold over a period of time by the screw in order to compensate the polymer shrinkage (packing stage), Fig. 1b. During the cooling stage, Fig. 1c, the cavity cools and the material solidifies. At the same time the screw moves backward, a new portion of the material in a granular form is inserted into the barrel from the hopper. Within the heated barrel the material is melted and homogenized by the rotating screw. After the cooling of the cavity the mold opens and the part is ejected, Fig. 1d. The following step is the mold closing, and the beginning of the next cycle. For a detailed description of the injection molding machine, including its various modifications, we refer to the monographs [29, 30].

The fiber orientation microstructure is primarily formed during the filling stage and remains unchanged after the solidification. The initial orientation of fibers may be considered to be random as the polymer melt, homogenized within the barrel, is inserted into the cavity. During the filling stage, the flow of the viscous polymer melt translates and rotates the suspended particles. The micrographs of cross sections of injection molded parts, e.g. [8, 36], show that the orientation of fibers exhibits a layered structure. In the mid-surface layer (core layer) the fibers are aligned dominantly perpendicular to the flow direction. On the other hand, in the layers neighboring upon the side walls (skin layers) the fibers lie dominantly parallel to the flow. In addition, the outer surface layers (shell layers) are usually detected having lower fiber concentration and random fiber orientation. As indicated in many works, e.g. [8, 24, 40], such a microstructure of fiber orientation has a correspondence to the flow behavior of the melt.

During the filling, an unsteady, non-isothermal flow with a moving free surface is observed within a geometrically complex cavity. In order to explain the suspension flow qualitatively let us

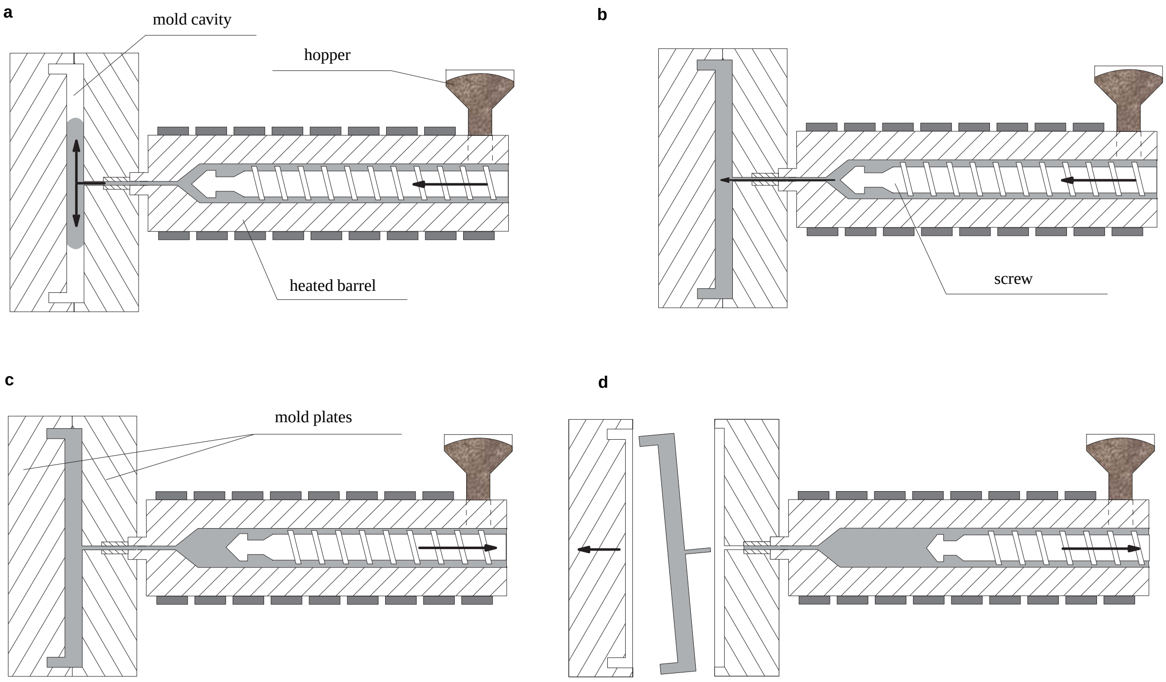

Figure 1. Basic stages of the injection molding processing cycle. **a** filling stage, **b** packing stage, **c** cooling stage, **d** ejection, for details see [29, 30]

consider an example of a radial, laminar flow between two parallel plates. The sketch, presented in Fig. 2, and the following comments are partly based on the results of filling simulations for a center-gated disk published in [3, 7, 35, 40] as well as on observations of the flow induced fiber microstructure [8, 40]. Because the thickness of the cavity is usually much smaller than other dimensions, one can formally separate the three flow regions [7]: the gate flow region, the lubrication region and the flow front region. Within the assumed lubrication region, Fig. 2, the velocity component in the thickness direction is negligible and the flow may be considered as two-dimensional. Assuming the parabolic velocity profile, one can estimate the kinematics of motion of particles. In the mid-surface layer of the lubrication region the elements of fluid undergo the stretching flow with the maximum strain rate in circumferential direction. Therefore, one can expect that a fiber inserted into the cavity with an arbitrary orientation, will be aligned in the direction perpendicular to the flow. On the other hand, the elements of fluid neighboring the cavity walls are exposed to the shear flow in the planes of radial cross sections. This shearing motion will align the fibers in the radial direction.

The situation is much more complicated in the neighborhood of the flow front. Firstly, as documented in many works, e.g. [7, 14], the behavior in the free surface region is governed by the fountain flow, which translates the elements of the fluid from the core zone towards the cavity walls. Secondly, the lower temperature of the cavity walls leads to the formation of the frozen layer (no-flow layer) behind the free surface, Fig. 2. The frozen layer propagates towards the flow front. Particles that enter the frozen layer will be fixed and unaffected by the flow. Thirdly, in the free surface zone one can expect a fiber concentration lower then elsewhere within the flow

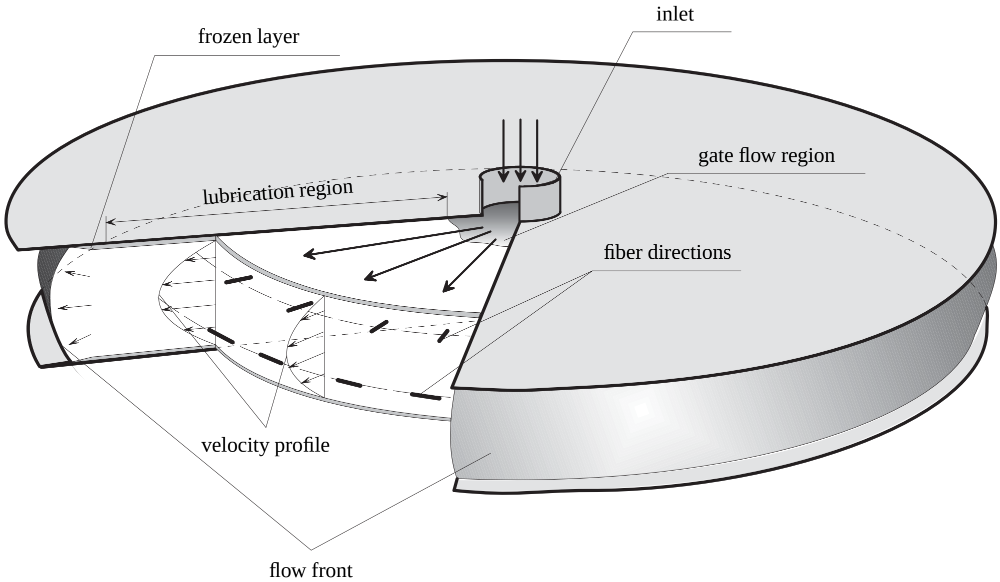

Figure 2. Sketch of a radial laminar flow between two parallel plates

domain. The flow behavior in the free surface region leads to the formation of the surface layers with lower fiber concentration and random fiber orientation. The discussed processes provide an explanation of principal mechanisms responsible to formation of the short-fiber microstructure. Further details regarding the flow behavior in complex cavities, e.g. the influence of abrupt changes of geometry of the cavity, formation of knit lines, etc., are discussed in the reviewing papers [14, 15, 39] among others.

In order to focus on the theoretical background of the filling process let us summarize some important features of the flow behavior:

- the flow is non-isothermal with phase transitions,
- the flow is non-steady with a free surface,
- the average fiber volume fraction of suspended particles lies within the range of 15–40%. The local concentration of fibers is affected by the flow and may vary within the flow domain. Therefore, the commonly used concepts of dilute, semi-dilute or concentrated suspensions are in general not suitable for the description of real processes, and
- the mold cavities are usually thin, so that the mold walls have essential influence on the fiber motion.

These factors may have an important influence on the formation of the flow-induced fiber microstructure and should be considered in a theory which allows a description of the filling stage. However, as far as we know, a general theory which is able to consider all the above features of the suspensions flow does not exist at present. The aim of this paper is to discuss the theoretical concepts for the prediction of the fiber orientation microstructure. Firstly, we give a brief review of the models recently proposed for the description of flow of fiber suspensions. Secondly, we develop and discuss a novel theory which treats the filling process as a phase-transitional flow of a binary medium.

1.2 Modelling of the Flow Induced Fiber Microstructure. State of the Art

The motion of an ellipsoidal particle immersed in a viscous fluid was firstly considered by G.B. Jeffery in [26]. The result obtained by Jeffery is most frequently used in the literature on suspended fluids. However, if applying it to injection molding simulations one should take into account a number of important restrictions. Therefore, without discussion regarding the Jeffery solution procedure, let us recall the main result obtained by Jeffery. In what follows we use the so-called direct tensor calculus which is conventional in many books on mechanics and rheology, e.g. [5, 21, 28, 32, 38, 45] among others. That means that the primary object is a vector **a** rather than a triple of numbers (coordinates). A second rank tensor **A** is any finite sum of the pairs of vectors $\mathbf{A} = \mathbf{a} \otimes \mathbf{b} + \ldots + \mathbf{c} \otimes \mathbf{d}$. If it is desirable, one can introduce a basis $\mathbf{g}_i$. In this case $\mathbf{a} = a^i \mathbf{g}_i$, $\mathbf{A} = \left(a^i b^j + \ldots + c^i d^j\right) \mathbf{g}_i \otimes \mathbf{g}_j$. Below we prefer to operate with vectors and tensors rather than with their coordinates a^i, $A^{ij} = a^i b^j + \ldots + c^i d^j$.

The undisturbed flow (the flow without particle) is supposed by Jeffery to satisfy the following restrictions

$$\nabla \mathbf{V}_0(\mathbf{x}) \equiv \boldsymbol{\Lambda} = \text{const}, \quad \text{tr}(\boldsymbol{\Lambda}) \equiv \nabla \cdot \mathbf{V}_0(\mathbf{x}) = 0, \quad \Rightarrow \quad \mathbf{V}_0 = \mathbf{r}_0 \cdot \boldsymbol{\Lambda}, \quad \boldsymbol{\Lambda} = \mathbf{d} - \boldsymbol{\phi} \times \mathbf{E},$$
$$\mathbf{d} \equiv \frac{1}{2}\left(\nabla \mathbf{V}_0 + \nabla \mathbf{V}_0^T\right), \ \text{tr}(\mathbf{d}) = 0, \ \boldsymbol{\phi} \equiv \frac{1}{2} \nabla \times \mathbf{V}_0, \quad \mathbf{V}_0 \cdot \nabla \mathbf{V}_0 = \mathbf{0} \quad \text{or} \quad \boldsymbol{\Lambda} \cdot \boldsymbol{\Lambda} = \mathbf{0}, \quad (1)$$

where $\mathbf{V}_0$ is the fluid velocity, **E** is the identity tensor, ∇ is the nabla operator,

$$\mathbf{r}_0 = x_k \, \mathbf{i}'_k, \quad \mathbf{i}'_k \cdot \mathbf{i}'_s = \delta_{ks},$$

and the orthonormal unit basis vectors $\mathbf{i}'_k$ are fixed in the reference system. The last two restrictions in Eqs. (1) are not given in Jeffery's work in the explicit form. However, they follow immediately from the Navier–Stokes equations

$$-\underline{\nabla p} + \mu \nabla \cdot \boldsymbol{\Lambda} = \rho \left(\frac{\partial \mathbf{V}_0}{\partial t} + \underline{\mathbf{r}_0 \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\Lambda}} \right) \quad \Rightarrow \quad p = p_0 = \text{const}, \quad \mathbf{r}_0 \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\Lambda} = \mathbf{0} \qquad (2)$$

with p as the pressure, μ as the fluid viscosity and ρ as the fluid density. Only the underlined terms in Eqs. (2) are not identically zero. Taking into account that the pressure p must be limited in the space we obtain the above mentioned restrictions. Let us define the ellipsoidal particle by means of the second rank tensor $\mathbf{A}_0$

$$\mathbf{A}_0 = a^2 \mathbf{i}'_1 \otimes \mathbf{i}'_1 + b^2 \mathbf{i}'_2 \otimes \mathbf{i}'_2 + c^2 \mathbf{i}'_3 \otimes \mathbf{i}'_3 \quad \Rightarrow \quad \mathbf{A}_0^{-1} = a^{-2} \mathbf{i}'_1 \otimes \mathbf{i}'_1 + b^{-2} \mathbf{i}'_2 \otimes \mathbf{i}'_2 + c^{-2} \mathbf{i}'_3 \otimes \mathbf{i}'_3,$$

where the numbers a, b, c are the semi-axes of the ellipsoid. Let $\mathbf{i}_k$ with $\mathbf{i}_k \cdot \mathbf{i}_s = \delta_{ks}$ be a triplet of the orthogonal unit basis vectors rigidly connected with the particle. Let us introduce the rotation tensor $\mathbf{P}(t)$ and the angular velocity vector $\boldsymbol{\omega}(t)$ of the ellipsoidal particle by

$$\mathbf{P}(t) \equiv \mathbf{i}_k(t) \otimes \mathbf{i}'_k, \quad \dot{\mathbf{P}}(t) = \boldsymbol{\omega}(t) \times \mathbf{P}(t) \quad \Rightarrow \quad \dot{\mathbf{m}}(t) = \boldsymbol{\omega}(t) \times \mathbf{m}(t), \qquad (3)$$

where $\mathbf{m}(t) \equiv \mathbf{P}(t) \cdot \mathbf{m}_0$ and $\mathbf{m}_0$ is an arbitrary vector fixed with respect to the ellipsoid $\mathbf{A}_0$. In this case the rotating ellipsoid is determined by the tensor

$$\mathbf{A}(t) = \mathbf{P}(t) \cdot \mathbf{A}_0 \cdot \mathbf{P}^T(t). \qquad (4)$$

The surface S of the rotating ellipsoid is determined by the equation

$$\mathbf{r} \cdot \mathbf{A}^{-1}(t) \cdot \mathbf{r} = 1, \quad \mathbf{r} = x_k \, \mathbf{i}_k, \quad \mathbf{r}(t) = \mathbf{P}(t) \cdot \mathbf{r}_0. \tag{5}$$

Jeffery considered the following problem: find the solution of quasi-static Navier–Stokes equations

$$-\nabla p + \mu \triangle \mathbf{V} = \mathbf{0}, \tag{6}$$

satisfying the boundary conditions

$$\mathbf{V}(t)|_S = \boldsymbol{\omega}(t) \times \mathbf{r}_S(t), \quad \text{and} \qquad \mathbf{V}(t) \, \rightarrow \, \mathbf{V}_0 \quad \text{when} \quad |\mathbf{r}| \rightarrow \infty. \tag{7}$$

Jeffery solved Eqs. (6)–(7) and calculated the moment acting on the particle. Setting this moment equal to zero Jeffery determined the angular velocity $\boldsymbol{\omega}$ of the particle in the following form

$$\boldsymbol{\omega} = \boldsymbol{\phi} + \big(\mathrm{tr}(\mathbf{A})\, \mathbf{E} - \mathbf{A}\big)^{-1} \cdot (\mathbf{A} \cdot \mathbf{d})_\times \,, \quad (\mathbf{a} \otimes \mathbf{b})_\times \equiv \mathbf{a} \times \mathbf{b}. \tag{8}$$

This is the invariant form of Eqs. (36) presented in the Jeffery paper. Let us emphasize that the constant vector $\boldsymbol{\phi}$ and the constant tensor $\mathbf{d}$ are defined in terms of the vector $\mathbf{V}_0$ by means of Eqs. (1). However, the tensor $\mathbf{A}$ contains the unknown rotation tensor $\mathbf{P}(t)$. In order to find $\mathbf{P}(t)$ one has to solve the left Darboux problem, see e.g. [43]

$$\dot{\mathbf{P}} = \Big[\boldsymbol{\phi} + \big(\mathrm{tr}(\mathbf{A})\, \mathbf{E} - \mathbf{A}\big)^{-1} \cdot (\mathbf{A} \cdot \mathbf{d})_\times\Big] \times \mathbf{P}.$$

If we multiply this equation by the vector $\mathbf{m}_0$, then we can rewrite the Jeffery result for an arbitrary vector $\mathbf{m}(t)$ rigidly connected with the ellipsoid

$$\dot{\mathbf{m}}(t) = \Big[\boldsymbol{\phi} + \big(\mathrm{tr}(\mathbf{A})\, \mathbf{E} - \mathbf{A}\big)^{-1} \cdot (\mathbf{A} \cdot \mathbf{d})_\times\Big] \times \mathbf{m}(t). \tag{9}$$

A more familiar form of Eq. (9) can be obtained in the case when the tensor $\mathbf{A}(t)$ is transversely isotropic and the vector $\mathbf{m}(t)$ is the axis of symmetry of $\mathbf{A}$. If $a = b \neq c$ then we have

$$\mathbf{A} = c^2 \mathbf{m} \otimes \mathbf{m} + a^2 \left(\mathbf{E} - \mathbf{m} \otimes \mathbf{m}\right) \Rightarrow \big(\mathrm{tr}(\mathbf{A})\, \mathbf{E} - \mathbf{A}\big)^{-1} = \frac{1}{2a^2} \mathbf{m} \otimes \mathbf{m} + \frac{1}{c^2 + a^2} \left(\mathbf{E} - \mathbf{m} \otimes \mathbf{m}\right),$$

$$(\mathbf{A} \cdot \mathbf{d})_\times = (c^2 - a^2) \mathbf{m} \times \mathbf{d} \cdot \mathbf{m}, \quad \big(\mathrm{tr}(\mathbf{A})\, \mathbf{E} - \mathbf{A}\big)^{-1} \cdot (\mathbf{A} \cdot \mathbf{d})_\times = \frac{c^2 - a^2}{c^2 + a^2} \mathbf{m} \times \mathbf{d} \cdot \mathbf{m}.$$

With these relations Eq. (9) can be rewritten as follows

$$\dot{\mathbf{m}} = \left(\boldsymbol{\phi} + \frac{c^2 - a^2}{c^2 + a^2} \mathbf{m} \times \mathbf{d} \cdot \mathbf{m}\right) \times \mathbf{m}. \tag{10}$$

Using Eq. (1) and eliminating the vector $\boldsymbol{\phi}$ from Eq. (10) we obtain the most popular form of the Jeffery result

$$\dot{\mathbf{m}} = (\mathbf{d} - \boldsymbol{\Lambda}) \cdot \mathbf{m} + \frac{c^2 - a^2}{c^2 + a^2} \left(m^2 \mathbf{d} \cdot \mathbf{m} - (\mathbf{m} \cdot \mathbf{d} \cdot \mathbf{m}) \mathbf{m}\right). \tag{11}$$

with $m = |\mathbf{m}|$. If $m = 1$ then instead of Eq. (11) we have

$$\dot{\mathbf{m}} = (\mathbf{d} - \mathbf{\Lambda}) \cdot \mathbf{m} + \frac{c^2 - a^2}{c^2 + a^2} (\mathbf{d} \cdot \mathbf{m} - (\mathbf{m} \cdot \mathbf{d} \cdot \mathbf{m})\mathbf{m}), \quad \mathbf{\Lambda} \equiv \nabla \mathbf{V}_0, \quad 2\mathbf{d} = \mathbf{\Lambda} + \mathbf{\Lambda}^T. \qquad (12)$$

This is exactly the form of the Jeffery equation used in many works on fiber suspensions, e.g. [1, 4, 3, 15, 25, 39]. Note that the vector $\mathbf{V}_0$ satisfies the very strong restrictions given in Eq. (1). It was not proved wether the equation (12) can be used in other cases. To be correct we have to mark that Eqs. (9)-(12) are not present in the Jeffery paper. However, the necessary theoretical background for the derivation of these equations was well-known at the beginning of the XIX century. In derivation of Eqs. (9)-(12) the only essential result is the expression (8) for the angular velocity which was found by Jeffery. Thus we may consider Eqs. (9)-(12) as the Jeffery equations.

Now we are able to discuss the applications of the Jeffery result in the literature on the subject under consideration. Firstly, instead of Eqs. (12) the following equations are used

$$\dot{\mathbf{m}} = (\mathbf{d} - \mathbf{\Lambda}) \cdot \mathbf{m} + \frac{c^2 - a^2}{c^2 + a^2} (\mathbf{d} \cdot \mathbf{m} - (\mathbf{m} \cdot \mathbf{d} \cdot \mathbf{m})\mathbf{m}), \quad \mathbf{\Lambda} \equiv \nabla \mathbf{V}, \quad 2\mathbf{d} = \mathbf{\Lambda} + \mathbf{\Lambda}^T, \qquad (13)$$

where the vector $\mathbf{V}$ is assumed to be the actual flow velocity without any restriction, except $\nabla \cdot \mathbf{V} = 0$. Secondly, it is supposed that the solution of Eq. (13) with the initial condition $|\mathbf{m}(0)| = 1$ is a unit vector, see, for example, p. 257 of [25]. However, it is easy to prove that this assumption is not valid for any case. The restriction $\mathbf{m} \cdot \mathbf{m} = 1$ must be connected with the Eq. (13) as an additional condition.

In the pioneering works on the injection molding simulations, e.g. [40], the Jeffery equation is numerically integrated for the known velocity field in order to calculate the fiber directions. The velocity gradient is computed by solving the flow problem of a Newtonian fluid. Such an approach is based on the assumption that interactions between the particles are negligible. In [39], p.165, Tucker and Advani pointed out that "the interaction between the multiple particles appears to be the most significant "non-Jeffery" effect in practical composite material problems". The common approach in modelling of the filling process is to treat the flow of a fiber suspension as the flow of a single-component anisotropic fluent medium, e.g. [15, 39]. Following this approach, the main problem is to find a rheological equation connecting the stress generated by the motion of the fluid with local characteristics of the motion. For a viscous incompressible fluid Batchelor [6] introduced the following equation

$$\boldsymbol{\sigma} = -p\mathbf{E} + \boldsymbol{\mu} \cdot\cdot \mathbf{\Lambda}, \quad \mathbf{\Lambda} = \nabla \mathbf{V}, \qquad (14)$$

where $\boldsymbol{\sigma}$ is the stress tensor and $\boldsymbol{\mu}$ is the fourth rank viscosity tensor determined by the local state of the fluid. Various approaches have been proposed in order to find a particular form of the constitutive equation. Batchelor [6] discussed the volume averaging procedure for a suspension. An important point in his consideration is the assumption that the inertia forces associated with fluctuations about the average motion are small if compared with the viscous forces and that the equation of the motion of a fluid reduces to the linear quasi-static Stokes equation. He found the following expression for the bulk stress in a suspension

$$\boldsymbol{\sigma} = -p\mathbf{E} + \mu(\mathbf{\Lambda} + \mathbf{\Lambda}^T) + \boldsymbol{\sigma}_p \qquad (15)$$

In order to obtain the particle stress $\boldsymbol{\sigma}_p$ in Eqs. (15) one should calculate the local velocity and stress fields of the fluid around a particle. Batchelor discussed the averaging procedure for dilute suspensions, i.e. assuming that the flow around each particle is unaffected by the presence of others. Based on the Jeffery solution for an ellipsoidal particle Batchelor obtained an explicit expression for $\boldsymbol{\sigma}_p$. Finally, he calculated the viscosity tensor $\boldsymbol{\mu}$ in Eq. (14) for the case of perfectly aligned particles for a given orientation state as well as for randomly oriented particles by means of averaging over all orientations.

The state of art on rheology of fiber suspensions can be found in reviews [15, 33, 39]. In the literature on rheology one distinguishes between dilute, semi-dilute and concentrated suspensions. Assuming that the immersed particles are slender bodies of revolution with a and c as particle dimensions, $a_p = c/a > 1$ as the particle aspect ratio and ξ is the fiber volume fraction, one specifies the suspension to be dilute when $\xi a_p^2 < 1$; semi-dilute when $1 < \xi a_p^2 < a_p$ and concentrated when $\xi a_p^2 > a_p$. This classification is made with regard to the kind of interactions between the particles by flow of fiber suspension, e.g. [15, 39]. In the first case one assumes no interactions, in the second case one assumes the interaction of the hydrodynamic nature and in the third case the interactions may have both hydrodynamic and direct mechanic origins. In fact, the concepts of dilute or non-dilute suspensions are intuitive assumptions rather then approximations of any general constitutive model. Such a model does not exist. The only known fact is that the commercial materials are non-dilute, see, e.g. [39].

In order to account for the fiber-fiber interactions as well as to consider the randomness of the fiber orientation at the inlet zone, the commonly used approach is the orientational averaging. As an example, let us introduce the model proposed by Dinh and Armstrong [12]. The starting point is the expression (15) for the bulk stress. In order to avoid the evaluating of the local flow fields around each particle, the authors considered a single slender body test particle in an effective continuous medium. They used the orientational probability density function $\psi(\mathbf{m})$, where $\mathbf{m}$ is the unit vector associated with a test particle. Further, the influence of the surrounding media on the test particle is considered by means of a surface force, which is determined from the transversely isotropic drag law. The resulting expression can be formulated as follows

$$\boldsymbol{\sigma} = -p\mathbf{E} + \mu(\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^T)\cdot\cdot\left[{}^{(4)}\mathbf{E} + \frac{nl^2}{12\mu}\zeta_p\mathbf{a}_4\right], \quad \mathbf{a}_4 = \int\limits_{(S)} \Psi(\mathbf{m})\mathbf{m}\otimes\mathbf{m}\otimes\mathbf{m}\otimes\mathbf{m}dS. \quad (16)$$

Here n is the number of particles per unit volume, l is the particle length, and ζ_p is the drag coefficient determined by

$$\zeta_p = \frac{2\pi\mu l}{\ln(2h/d)}$$

with d as the fiber diameter and $h = (nl)^{-1/2}$ for aligned systems while $h = (nl^2)^{-1}$ for random systems. In (16) ${}^{(4)}\mathbf{E}$ is the fourth rank identity tensor and $\mathbf{a}_4$ denotes the fourth rank structure tensor, which characterizes the actual fiber orientation state and dS is a differential element on a unit sphere. In order to formulate the evolution equation for the structure tensor one needs an equation for the probability density function. Assuming that the mechanism of fiber-fiber interactions is governed by the rotary Brownian motion, e.g., [39] the Smoluchowski type equation, e.g. [9, 13], is applied

$$\dot{\Psi} + \boldsymbol{\nabla}_s\cdot(\Psi\boldsymbol{\omega} - D_r\boldsymbol{\nabla}_s\Psi) = 0, \quad \boldsymbol{\omega} = \mathbf{m}\times\dot{\mathbf{m}}, \quad (17)$$

where $\dot{(\ldots)}$ denotes the material derivative, D_r is the coefficient of rotary diffusion, and

$$\boldsymbol{\nabla}_s(\ldots) = \mathbf{e}_k\epsilon_{ijk}m_i\frac{\partial(\ldots)}{\partial m_j}, \quad \mathbf{m}\cdot\mathbf{m} = 1$$

with ϵ_{ijk} as the permutation symbol. For the modelling of flow of fiber suspensions Eq. (17) is modified assuming that D_r is a scalar valued function of $\mathbf{d}$, e.g. [39]. Furthermore, $\boldsymbol{\omega}$ in Eq. (17) is treated as the angular velocity of a single particle. Following [1] the Jeffery's result (11) is inserted into Eq. (17). Using the series representation of $\Psi(\mathbf{m})$ by means of the spherical harmonics and introducing the moments of $\Psi(\mathbf{m})$ by

$$\mathbf{a}_n = \int\limits_{(S)} \Psi(\mathbf{m})\mathbf{m}^{\otimes n} dS, \quad n = 2, 4, \ldots,$$

where $\mathbf{a}_n$ are termed as n-th rank structure tensors and $(\ldots)^{\otimes n}$ is the n-th dyadic product, Eq. (17) is replaced by a set of coupled evolution equations for $\mathbf{a}_n$ [1]. In the injection molding simulations the evolution equation for $\mathbf{a}_2$

$$\dot{\mathbf{a}}_2 = (\mathbf{a}_2 \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{a}_2) + \lambda(\mathbf{d} \cdot \mathbf{a}_2 + \mathbf{a}_2 \cdot \mathbf{d} - 2\mathbf{a}_4 \cdot\cdot \mathbf{d}) - 6D_r \left(\mathbf{a}_2 - \frac{1}{3}\mathbf{E}\right), \quad \mathbf{w} = \boldsymbol{\Lambda} - \mathbf{d} \quad (18)$$

is usually solved while for $\mathbf{a}_4$ a closure approximation is applied. Various types of closure approximations can be found in [1, 2, 11, 15, 31]. Let us note that the model (16) in connection with the evolution equation of the type (18) is widely used in injection molding simulations, e.g. [4, 10]. The tensor $\mathbf{a}_2$ (and in some cases $\mathbf{a}_4$) provides the information about the actual state of the fiber orientation.

Finally, let us discuss the anisotropic fluid models developed within the framework of the continuum mechanics. Based on the invariance conditions Ericksen [17] found a simplest form of the constitutive model for a transversely isotropic single-component incompressible fluid

$$\begin{aligned} \boldsymbol{\sigma} &= -p\mathbf{E} + 2\mu\mathbf{d} + (\mu_1 + \mu_2 \mathbf{m} \cdot \mathbf{d} \cdot \mathbf{m})\mathbf{m} \otimes \mathbf{m} + 2\mu_3[\mathbf{m} \otimes (\mathbf{d} \cdot \mathbf{m}) + (\mathbf{d} \cdot \mathbf{m}) \otimes \mathbf{m}], \\ \overset{\circ}{\mathbf{m}} &\equiv \dot{\mathbf{m}} - \mathbf{w} \cdot \mathbf{m} = \lambda(\mathbf{d} \cdot \mathbf{m} - (\mathbf{m} \cdot \mathbf{d} \cdot \mathbf{m})\mathbf{m}), \quad \mathbf{m} \cdot \mathbf{m} = 1. \end{aligned} \quad (19)$$

Here λ and μ, μ_1, μ_2, μ_3 are constants, $\mathbf{m}$ is a unit vector and $\overset{\circ}{(\ldots)}$ denotes the co-rotational time derivative. The concept of Ericksen's fluid associates the anisotropic behavior of a fluid with a director $\mathbf{m}$, changing with time according to the second equation in (19). Let us note that the second equation in Eqs. (19) formally coincides with the Jeffery Eq. (13) for the single particle by setting $\lambda = (c^2 - a^2)/(c^2 + a^2)$. However, it should be noted that Eqs. (19) and (13) are derived based on two different considerations. Furthermore, the constitutive model (19) must be introduced together with conservation laws in continuum mechanics [16]. Eringen [18, 19] developed a micro-polar theory of anisotropic fluids and applied it to the flow of fiber suspensions. The important feature of his theory is the modified balance law for the inertia tensor which accounts the phenomenon of sticking of the fluid to suspended particles. Assuming the inertia tensor to be transversely isotropic Eringen derived the evolution equation (Eq. (6.4) in [18]) which is similar to Eqs. (13) and (19).

In the last two decades a large amount of work has been directed to simulations of the injection molding process. For an overview of existing models and their numerical realizations we refer to [14]. Furthermore, various commercial software have been developed, e.g. Moldflow® [27], which have a purpose to simulate the whole injection molding cycle and to optimize the process conditions. Some years ago the commercial filling software have been extended by the units allowing the prediction of fiber orientation microstructure. These units are based on the above discussed rheological equations of state and compute the fiber orientation by means of structure tensors. The present state of art on modelling the flow of fiber suspensions is given in [15, 39], where a number of open questions is discussed. Regarding the theoretical approach one may note that there is no unified concept in modelling of suspended fluids and as pointed out in [19], p. 117, “at present there is no agreement on any one particular theory”.

1.3 The scope of the paper

In order to underline the purposes of our approach let us introduce the following definitions. Let $\eta_1(\mathbf{x}, t)$ be the density of fluid particles and $\eta_2(\mathbf{x}, t)$ be the density of rigid particles at a given point $\mathbf{x}$ of an inertial reference system. Specifying by $d\,N_1$ and $d\,N_2$ the number of particles for the first and the second components in a control volume $d\,V$ we can write

$$d\,N_1 = \eta_1(\mathbf{x}, t)d\,V, \quad \eta_1 \geq 0; \quad d\,N_2 = \eta_2(\mathbf{x}, t)d\,V, \quad \eta_2 \geq 0.$$

The functions $\eta_1(\mathbf{x}, t)$ and $\eta_2(\mathbf{x}, t)$ are the principal unknowns in the theory of mixtures. The function $\eta_2(\mathbf{x}, t)$ is particularly important, because it characterizes the distribution of the rigid particles in a fluid. This distribution will affect the final mechanical properties of the material after the processing. A strong difference between the properties leads to the necessity to introduce different models for each of the two components. The first one is a set of fluid particles characterized by the particle density η_1 and the mass density ρ_1. In what follows this constituent will be termed as the liquid component. The second component is a set of rigid particles immersed in the viscous fluid with η_2 and ρ_2 as the particle and mass densities. Below, by making constitutive assumptions, we shall exclude the possibility that the rigid particles may form a solid body. Thus, we shall assume that the constituent of particles-fibers behaves like a liquid. In what follows the second component will be termed as the solid-liquid component.

The liquid component may be considered as a viscous fluid with some additional properties. For example, the stress tensor of the fluid in our model will be nonsymmetric. For the solid-liquid component it is necessary to consider not only the translation motion but also the rotations. Multi-component mixtures were studied in many works, e.g. [20, 34] (see also works cited therein). In the physico-chemical hydrodynamics the diffusion processes play an essential role [34]. Namely, the diffusion determines the relative velocities of constituents by means of the Fick laws. In our case these velocities are determined by the external conditions and by the viscous properties of the fluid. The diffusion can be neglected.

In what follows let us briefly describe the framework of the paper and the distinctive features of our approach:

1. We will assume the particle density and the mass density as independent functions in order to take into account the phenomena of sticking of the fluid particles to the rigid particles. Consequently, the particle balance equations and the mass balance equations are independent from each other.

2. The liquid constituent will be supposed to be a non-polar viscous fluid, but with the non-symmetric stress tensor.

3. The state of the solid-liquid constituent will be described by means of the antisymmetric stress tensor and of the antisymmetric moment stress tensor.

4. Let the vectors $\mathbf{V}_1(\mathbf{x}, t)$ and $\mathbf{V}_2(\mathbf{x}, t)$ be the velocities of the particles of liquid and solid-liquid components, respectively. We will assume that $\mathbf{V}_1(\mathbf{x}, t) \neq \mathbf{V}_2(\mathbf{x}, t)$, i.e. in our approach, the constituents may slide with respect to each other. Therefore, the forces of friction between components will be taken into account.

5. The equations of motion are given for open physical systems and contain additional terms which are responsible for the phenomenon of sticking.

6. The chemical potential is introduced on the base of the so-called reduced equation of the energy balance. Our definition differs from the definitions proposed in [20, 34]. By neglecting the sticking of the fluid particles to fibers, the chemical potential is conserved.

7. We assume that the constituents may have different temperatures. Such a difference may be important if we want to produce the material with desired mechanical properties. Thus, the temperature fields are in general discontinuous. In such a case the conventional form of the second law of thermodynamics, for example the Clausius-Duhem inequality, is not applicable. So, we give an alternative statement of the second law of thermodynamics as a set of two inequalities. The problem connected with the modelling of the heat exchange is discussed.

8. The main purpose of our approach is to describe the real technological process in which the mixture has a stage of solidification. The solidification takes place not only at the final stage of the process, but also during filling near the cavity walls. Therefore, the models of suspensions, based on the assumption of the anisotropic incompressible viscous fluid, are in general not suitable. In this paper we will discuss a model for compressible fluid with phase transitions of the liquid-solid type. The phase transitions are described by means of a proposed constitutive equation for the pressure.

2 Kinematical relations

It seems to be evident that in general a continuous medium cannot be modelled as a smooth differentiable manifold. Indeed, as it is known from experience, the particles which are neighbors at a moment of time, do not necessarily occupy neighboring positions at any later time. In such a medium one can expect the occurrence of tangential discontinuities or the nucleation of cavities. For a multi-component medium the situation is more complicated since different components may glide on each other and interact, and the interaction takes place between different species. For instance, let A_1, B_1, $C_1, \ldots$ be some marked parts of a first component and A_2, B_2, $C_2, \ldots$ be the parts of the second one. Let us assume that at a moment of time the parts A_1 and A_2 interact. Then, at any other moment of time another two parts, for example A_1 and B_2 will contact. From this follows that the material description, which assumes the neighboring particles to be always neighbors, is in general not applicable for multi-component media. The only possible way to formulate a theory for such the media is the use of the pure spatial description. That means that in contrast to the material description, all principal operators in the theory must be directly defined within the reference frame rather then over a differentiable manifold. The introduction of these operators is the purpose of the following considerations.

Let us introduce a control volume in the reference frame and assume that the volume at the time t is filled by the medium. The medium may move with respect to the reference frame, or the medium may be at rest and the reference frame may move with respect to the medium. The difference between these two situations is not essential from the kinematical point of view. The important role plays the velocity field $\mathbf{V}(\mathbf{x}, t)$, where the vector $\mathbf{x}$ defines a point of the reference frame. Thus, the vector $\mathbf{V}(\mathbf{x}, t)$ characterizes the velocity of that particle which at the given moment of time t occupies the point $\mathbf{x}$. Let $\mathbf{K}(\mathbf{x}, t)$ be a given field, which can be a tensor of any rank. This field describes a physical quantity of that particle, which is placed at the point $\mathbf{x}$ of the reference frame at the given moment of time. Let us use the following definition of the material derivative [44]:

The material derivative of a quantity $\mathbf{K}(\mathbf{x}, t)$ *is the limit of the fraction*

$$\frac{\delta}{\delta t}\mathbf{K}(\mathbf{x}, t) = \lim_{\triangle t \to 0} \frac{\mathbf{K}(\mathbf{x} + \triangle \mathbf{s}, t + \triangle t) - \mathbf{K}(\mathbf{x}, t)}{\triangle t}, \quad \triangle \mathbf{s} = \mathbf{V}(\mathbf{x}, t)\, \triangle t. \tag{20}$$

In this definition $\triangle \mathbf{s}$ (by neglecting the terms of the second and higher order of magnitude) is the way, passed within the time $\triangle t$ by that particle, which at the time t was placed at the point $\mathbf{x}$. The nominator in Eq. (20) can be rewritten by means of the following expansion

$$\mathbf{K}(\mathbf{x} + \triangle \mathbf{s}, t + \triangle t) = \mathbf{K}(\mathbf{x}, t + \triangle t) + \triangle \mathbf{s} \cdot \nabla \mathbf{K}(\mathbf{x}, t + \triangle t).$$

From the definition (20) follows

$$\frac{\delta}{\delta t}\mathbf{K}(\mathbf{x}, t) = \frac{d}{dt}\mathbf{K}(\mathbf{x}, t) + \mathbf{V}(\mathbf{x}, t) \cdot \nabla \mathbf{K}(\mathbf{x}, t). \tag{21}$$

In the first term of the right hand side in Eq. (21) one can formally replace the total time derivative by the partial one. However, such a replacement may lead to difficulties by a change of the reference frame. In several situations the above definition of the material derivative may be not convenient, because the point of observation is assumed to be fixed. Within the conventional Euler's description this is always the case. However, by a change of the reference frame one needs to consider a moving point of observation. Therefore, let us introduce an extended definition of the material derivative. Let $\mathbf{y}(t)$ be a point of observation, which can move according to any given law. The velocity field $\mathbf{V}(\mathbf{x},\ t)$ is defined in those points of the reference frame, which are occupied by particles of the medium. Therefore, it is also defined in the points $\mathbf{y}(t)$. Let us accept the following modification of the definition (20)

$$\frac{\delta}{\delta t}\mathbf{K}(\mathbf{y}(t), t) = \lim_{\triangle t \to 0} \frac{\mathbf{K}(\mathbf{y}(t + \triangle t) + \triangle \mathbf{s}, t + \triangle t) - \mathbf{K}(\mathbf{y}(t), t)}{\triangle t}, \tag{22}$$

where

$$\triangle \mathbf{s} = \mathbf{V}_r(\mathbf{y}(t), t)\, \triangle t, \quad \mathbf{V}_r(\mathbf{y}(t), t) = \mathbf{V}(\mathbf{y}(t), t) - \frac{d\mathbf{y}(t)}{dt}.$$

Here the velocity $\mathbf{V}_r(\mathbf{y}(t), t)$ is the relative velocity of the material point with respect to the moving point $\mathbf{y}(t)$.

Now instead of (21) we have

$$\frac{\delta}{\delta t}\mathbf{K}(\mathbf{y}(t), t) = \frac{d}{dt}\mathbf{K}(\mathbf{y}(t), t) + \left(\mathbf{V}(\mathbf{y}(t), t) - \frac{d\mathbf{y}}{dt}\right) \cdot \nabla \mathbf{K}(\mathbf{y}(t), t). \tag{23}$$

The proposed definition (23) does not coincide with the conventional definition of the material derivative. However, one can rewrite Eq. (23) as follows

$$\frac{\delta}{\delta t}\mathbf{K}(\mathbf{y}(t), t) = \frac{\partial}{\partial t}\mathbf{K}(\mathbf{y}(t), t) + \mathbf{V}(\mathbf{y}(t), t) \cdot \nabla \mathbf{K}(\mathbf{y}(t), t). \tag{24}$$

The last expression just coincides with the commonly used material derivative, the difference is only $\mathbf{y}(t)$. However, in Eq. (24) the relative character of the velocity $\mathbf{V}_r(\mathbf{y}(t), t)$ is hidden. The expression (24) looks like the total derivative d/dt. Therefore, in the literature the notation d/dt is usually preferred. However, in the general case we have to distinguish between d/dt and $\delta/\delta t$. In fact, the total time derivative is

$$\frac{d}{dt}\mathbf{K}(\mathbf{y}(t), t) = \frac{\partial}{\partial t}\mathbf{K}(\mathbf{y}(t), t) + \frac{d\mathbf{y}(t)}{dt} \cdot \nabla\mathbf{K}(\mathbf{y}(t), t), \quad \frac{d\mathbf{y}(t)}{dt} \neq \mathbf{V}.$$

The last expression coincides with the material derivative if and only if the point of observation $\mathbf{y}(t)$ coincides with the position vector of a fixed particle. In the case of multi-components media such a situation is impossible because in a point of observation may be several different particles with different velocities.

For the material derivative all rules of differentiation are valid. For example

$$\frac{\delta}{\delta t}(\mathbf{a} \otimes \mathbf{b}) = \frac{\delta \mathbf{a}}{\delta t} \otimes \mathbf{b} + \mathbf{a} \otimes \frac{\delta \mathbf{b}}{\delta t}.$$

On the other hand, it is known that

$$\frac{d}{dt}\nabla = \nabla\frac{d}{dt}, \quad \frac{\delta}{\delta t}\nabla \neq \nabla\frac{\delta}{\delta t}.$$

In binary mixtures one usually assumes that one point of a reference frame can be simultaneously occupied by the particles of both species, e.g. [34]. The mixture considered in this work consists from two components. The first one includes particles of the viscous fluid. The second one is built up from small rigid bodies – fibers, which can be considered as ellipsoids of revolution. Let us introduce the notations: $\mathbf{V}_1(\mathbf{x},\ t)$ is the velocity vector of that particle of the fluid, which at the given time t is placed in $\mathbf{x}$ of the reference frame; $\mathbf{V}_2(\mathbf{x},\ t)$ is the velocity vector of that particle-fiber, which at the given time t occupies the place $\mathbf{x}$ of the reference frame. The vector $\mathbf{V}_2(\mathbf{x},\ t)$ will be treated as the velocity vector of the center of mass of a particle-fiber. Let us note that in theories of mixtures one usually assumes $\mathbf{V}_1(\mathbf{x},\ t) = \mathbf{V}_2(\mathbf{x},\ t)$ [34].

By taking the material derivative of the velocity vector $\mathbf{V}_1(\mathbf{x},\ t)$, we obtain the acceleration vector of a fluid particle

$$\mathbf{W}_1(\mathbf{x}(t), t) = \frac{d}{dt}\mathbf{V}_1(\mathbf{x}(t), t) + \left(\mathbf{V}_1(\mathbf{x}(t), t) - \frac{d\mathbf{x}(t)}{dt}\right) \cdot \nabla\mathbf{V}_1(\mathbf{x}(t), t).$$

The material derivative of the velocity vector $\mathbf{V}_2(\mathbf{x},\ t)$ yields the velocity of the center of mass of a particle-fiber

$$\mathbf{W}_2(\mathbf{x}(t), t) = \frac{d}{dt}\mathbf{V}_2(\mathbf{x}(t), t) + \left(\mathbf{V}_2(\mathbf{x}(t), t) - \frac{d\mathbf{x}(t)}{dt}\right) \cdot \nabla\mathbf{V}_2(\mathbf{x}(t), t).$$

Here we used the definition (23) assuming a moving point of observation. One can examine the difference between Eqs. (23) and (24) by calculating the acceleration vectors for the case

$$\mathbf{V} \quad \to \quad \mathbf{V} + \mathbf{V}_0, \quad \mathbf{V}_0 = \mathrm{const}.$$

Such a replacement is conventional, if one needs to use the Galilei relativity principle or to change the reference frame. It is clear that such a transformation should not change the accelerations.

This will be the case if we use Eq. (23). But when applying Eq. (24) one must be careful to avoid mistake.

Let us assume that the fluid particle and the particle-fiber, occupying at a given time t the point $\mathbf{x}$, at $t_0 \leq t$ were located at $\mathbf{x}_0$ and $\mathbf{x}_0^*$, respectively. The displacement vectors $\mathbf{u}_1(\mathbf{x}, t) = \mathbf{x} - \mathbf{x}_0$ and $\mathbf{u}_2(\mathbf{x}, t) = \mathbf{x} - \mathbf{x}_0^*$ are determined from the velocities by means of the following relations

$$\mathbf{V}_i(\mathbf{x}, t) = \frac{d}{dt}\mathbf{u}_i(\mathbf{x}, t) + \mathbf{V}_i(\mathbf{x}, t) \cdot \nabla \mathbf{u}_i(\mathbf{x}, t) \quad \Rightarrow \quad \frac{d}{dt}\mathbf{u}_i(\mathbf{x}, t) = \mathbf{V}_i(\mathbf{x}, t) \cdot \mathbf{g}_i(\mathbf{x}, t) \tag{25}$$

with the notations

$$\mathbf{g}_i(\mathbf{x}, t) \equiv (\mathbf{E} - \nabla \mathbf{u}_i(\mathbf{x}, t)), \quad \det \mathbf{g}_i(\mathbf{x}, t) > 0. \tag{26}$$

By calculating the gradient of both parts of the second equation in (25) and taking into account the permutability of the gradient operator and the total time derivative, we obtain

$$\frac{d}{dt}\nabla \mathbf{u}_i(\mathbf{x}, t) + \mathbf{V}_i \cdot \nabla\nabla \mathbf{u}_i(\mathbf{x}, t) = \nabla \mathbf{V}_i(\mathbf{x}, t) \cdot \mathbf{g}_i(\mathbf{x}, t) \quad \Rightarrow$$
$$\nabla \mathbf{V}_i(\mathbf{x}, t) = \left(\frac{d}{dt}\nabla \mathbf{u}_i(\mathbf{x}, t) + \mathbf{V}_i \cdot \nabla\nabla \mathbf{u}_i(\mathbf{x}, t)\right) \cdot \mathbf{g}_i^{-1}(\mathbf{x}, t). \tag{27}$$

Equations similar to (27) can be found in [32]. The last equation can be rewritten in the equivalent form

$$\nabla \mathbf{V}_i(\mathbf{x}, t) = -\left(\frac{d}{dt}\mathbf{g}_i(\mathbf{x}, t) + \mathbf{V}_i \cdot \nabla \mathbf{g}_i(\mathbf{x}, t)\right) \cdot \mathbf{g}_i^{-1}(\mathbf{x}, t). \tag{28}$$

Equations (28) will be used later for the formulation of the reduced energy balance equation.

Till now, we did not make any distinction between the fluid particles and the particles-fibers. Let us introduce the rotations of particles-fibers, which will define their orientation in the reference frame. The determination of this orientation is one of the main purposes of the theory. Let us presume that at each point $\mathbf{x}$ of the reference frame a triple $\mathbf{d}_k$ with $\mathbf{d}_k \cdot \mathbf{d}_m = \delta_{km}$ is given. Let us introduce the proper orthogonal tensor $\mathbf{P}(\mathbf{x}, t)$, which describes the rotation of the particle-fiber, located at the point $\mathbf{x}$ at the time t with respect to the triple $\mathbf{d}_k$. Further, let us calculate the angular velocity of the rigid particle. Within the framework of the rigid body dynamics one can apply the Poisson equation [43]

$$\frac{d}{dt}\mathbf{P} = \boldsymbol{\omega} \times \mathbf{P}, \tag{29}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of a point in the body. It is clear that the definition (29) is not applicable to our case, since at different instances of time the point $\mathbf{x}$ of the reference frame is occupied by different particles. Therefore, the time derivative in (29) cannot be treated as a characteristic of a particle. Instead of the definition (29) we have to use the following modification of the Poisson equation

$$\frac{d}{dt}\mathbf{P}(\mathbf{x}(t), t) + \left(\mathbf{V}_2(\mathbf{x}(t), t) - \frac{d\mathbf{x}(t)}{dt}\right) \cdot \nabla \mathbf{P}(\mathbf{x}, t) = \boldsymbol{\omega}(\mathbf{x}(t), t) \times \mathbf{P}(\mathbf{x}(t), t). \tag{30}$$

Here the subscripts for the rotation tensor and for the angular velocity are dropped since these quantities are defined for the particles-fibers only. Let us prove whether the definition (30) corresponds to our intuitive considerations. Consider two motions of the same particle and assume that these two motions have different translation parts, but the same rotations. Let $\mathbf{x}_A(t)$ and $\mathbf{y}_A(t)$ be two translations of the particle A so that

$$\mathbf{y}_A(t) = \mathbf{x}_A(t) + \mathbf{f}_A(t), \quad \mathbf{P}(\mathbf{x}_A, t) = \mathbf{P}(\mathbf{y}_A, t).$$

Making use of Eq. (23) it can be shown that $\boldsymbol{\omega}(\mathbf{x}_A, t) = \boldsymbol{\omega}(\mathbf{y}_A, t)$. That means that the angular velocity of the particle does not depend on its translation motion.

3 Particle balance and mass balance equations

Let us consider three different cases. In the first one, we assume that the total number of particles in both the components remains unchanged. It seems to be evident that such a strongly restricting assumption is not satisfied in the reality. In the second one, we suppose that the total number of fibers remains constant, while the mass of fibers may vary due to the sticking of the fluid particles to the fibers. In this case the number of fluid particles is changing, i.e. the density of fluid particles η_1 is not constant. On the other hand, the density of particles-fibers η_2 remains constant, while the mass density of particles-fibers is changing. Finally, in the third situation both the density of fluid particles and the density of particles-fibers are changing. That means that not only the fluid particles can stick to the fibers but also the fibers may stick to each other. The sticking of fibers may lead to the formation of grains-clusters, which must be treated as new particles. Evidently, the last case is more realistic for concentrated suspensions. It is difficult to verify, how important are the effects of sticking for the short-time filling processes. The quantitative influence of the sticking effects on the flow process seems to be not significant. Nevertheless, let us discuss all the three situations separately.

Liquid and solid-liquid constituents have constant compositions. Let V be a control volume in the reference frame and the boundary of V be a closed surface $S = \partial V$. Then, for each of the introduced species we can formulate the following particle balance equations

$$\frac{d}{dt}\int\limits_{(V)} \eta_1\,(\mathbf{x},t)\,dV = -\int\limits_{(S)} \eta_1 \mathbf{n}\cdot\mathbf{V_1} dS = -\int\limits_{(V)} \boldsymbol{\nabla}\cdot(\eta_1 \mathbf{V}_1)\,dV, \tag{31}$$

where $\mathbf{V}_1(\mathbf{x},\ t)$ is the velocity of fluid particles,

$$\frac{d}{dt}\int\limits_{(V)} \eta_2\,(\mathbf{x},t)\,dV = -\int\limits_{(S)} \eta_2 \mathbf{n}\cdot\mathbf{V}_2 dS = -\int\limits_{(V)} \boldsymbol{\nabla}\cdot(\eta_2 \mathbf{V}_2)\,dV, \tag{32}$$

where $\mathbf{V}_2(\mathbf{x},\ t)$ is the velocity of particles-fibers. Note that in the case of the moving point of observation one should replace $\mathbf{V}_i$ by $\mathbf{V}_i - \dfrac{d\mathbf{x}}{dt}$ in Eqs. (31) and (32).

In the local form Eqs. (31) and (32) can be written as

$$\frac{d\eta_1}{dt} + \boldsymbol{\nabla}\cdot(\eta_1 \mathbf{V}_1) = 0, \quad \frac{d\eta_2}{dt} + \boldsymbol{\nabla}\cdot(\eta_2 \mathbf{V}_2) = 0. \tag{33}$$

Analogously to Eqs. (33), we can formulate the mass balance equations

$$\frac{d\rho_1}{dt} + \boldsymbol{\nabla}\cdot(\rho_1 \mathbf{V}_1) = 0, \quad \frac{d\rho_2}{dt} + \boldsymbol{\nabla}\cdot(\rho_2 \mathbf{V}_2) = 0, \tag{34}$$

where ρ_1 and ρ_2 are mass densities of the liquid and the solid-liquid components, respectively.

Let us introduce the following notations for the material derivatives

$$\frac{\delta_1 f}{\delta t} \equiv \frac{df}{dt} + \left(\mathbf{V}_1 - \frac{d\mathbf{x}}{dt}\right) \cdot \nabla f, \quad \frac{\delta_2 f}{\delta t} \equiv \frac{df}{dt} + \left(\mathbf{V}_2 - \frac{d\mathbf{x}}{dt}\right) \cdot \nabla f, \tag{35}$$

where f is an arbitrary scalar function (or any tensor-valued function). With the introduced notations Eqs. (33) and (34) take the following form

$$\frac{\delta_1 \eta_1}{\delta t} + \eta_1 \nabla \cdot \mathbf{V}_1 = 0, \quad \frac{\delta_2 \eta_2}{\delta t} + \eta_2 \nabla \cdot \mathbf{V}_2 = 0, \tag{36}$$

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \nabla \cdot \mathbf{V}_1 = 0, \quad \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \nabla \cdot \mathbf{V}_2 = 0. \tag{37}$$

The liquid constituent has a variable composition. In this situation, the density of the liquid component in the selected reference frame may change not only due to the flow, but also as a consequence of the sticking of the fluid particles to the particles-fibers. The fluid particles connected to fibers cannot be considered as fluid particles anymore. They should be related to the mass of fibers. The particle density of fibers may only change with regard to the motion of particles-fibers.

The particle balance equation for the liquid component (31) should be modified as follows

$$\frac{d}{dt} \int\limits_{(V)} \eta_1 (\mathbf{x},t)\, dV = \int\limits_{(V)} \chi_1 (\mathbf{x},t)\, dV - \int\limits_{(S)} \eta_1 \mathbf{n} \cdot \mathbf{V_1} dS = \int\limits_{(V)} [\chi_1 (\mathbf{x},t) - \nabla \cdot (\eta_1 \mathbf{V}_1)]\, dV, \tag{38}$$

where the function χ_1 is the rate of production (destruction) of fluid particles at a point of the reference frame.

The particle balance equation for the fibers remains unchanged. Therefore, Eqs. (36) take now the form

$$\frac{\delta_1 \eta_1}{\delta t} + \eta_1 \nabla \cdot \mathbf{V}_1 = \chi_1, \quad \frac{\delta_2 \eta_2}{\delta t} + \eta_2 \nabla \cdot \mathbf{V}_2 = 0. \tag{39}$$

The equations of the mass balance should be modified as follows

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \nabla \cdot \mathbf{V}_1 = \chi_{1m}, \quad \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \nabla \cdot \mathbf{V}_2 = \chi_{2m}, \tag{40}$$

where the functions χ_{1m} and χ_{2m} characterize the rates of mass production (destruction) of fluid particles and particles-fibers, respectively. Because the total mass density $\rho = \rho_1 + \rho_2$ does not change, the equation of the mass balance for the considered binary medium can be written down in the integral form

$$\frac{d}{dt} \int\limits_{(V)} \rho (\mathbf{x},t)\, dV = - \int\limits_{(S)} \rho \mathbf{n} \cdot \mathbf{V}_m dS = - \int\limits_{(V)} \nabla \cdot (\rho \mathbf{V}_m)\, dV, \quad \rho \mathbf{V}_m = \rho_1 \mathbf{V}_1 + \rho_2 \mathbf{V}_2. \tag{41}$$

The local form of Eq (41) is

$$\frac{\delta_m \rho}{\delta t} + \rho \nabla \cdot \mathbf{V}_m = 0, \quad \frac{\delta_m f}{\delta t} \equiv \frac{df}{dt} + \left(\mathbf{V}_m - \frac{d\mathbf{x}}{dt}\right) \cdot \nabla f. \tag{42}$$

Here the point of observation $\mathbf{x}(t)$ is selected to be the same for both the components. If we add Eqs. (40) and then subtract from the result Eq. (41), we obtain

$$\chi_{2m} = -\chi_{1m}, \tag{43}$$

i.e. the amount of mass acquired per unit time by the solid-liquid component per unit time is equal to the amount of mass lost by the liquid component.

Liquid and solid-liquid components have variable compositions. The particle balance equations can be formulated according to the above discussed procedure

$$\frac{\delta_1 \eta_1}{\delta t} + \eta_1 \nabla \cdot \mathbf{V}_1 = \chi_1, \quad \frac{\delta_2 \eta_2}{\delta t} + \eta_2 \nabla \cdot \mathbf{V}_2 = \chi_2, \tag{44}$$

where the function χ_2 characterizes the production rate of particles-fibers. The mass balance equations remain the same

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \nabla \cdot \mathbf{V}_1 = \chi_{1m}, \quad \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \nabla \cdot \mathbf{V}_2 = \chi_{2m}, \quad \chi_{2m} = -\chi_{1m}. \tag{45}$$

Let us emphasize that the introduced particle densities and the mass densities are independent functions. Consequently, Eqs. (44) and (45) are independent. However, the functions χ_1 and χ_{1m} can be assumed to be connected by means of equation $\chi_{1m} = m\chi_1$, where m characterizes the mass of one fluid particle. The last assumption is evident, since the fluid particles cannot form clusters.

Eqs. (44) and (45) can be rewritten in a scalar form. From Eqs. (28) follows

$$\nabla \cdot \mathbf{V}_i(\mathbf{x}, t) = -\mathbf{g}_i^{-1}(\mathbf{x}, t) \cdot\cdot \left(\frac{\delta_i}{\delta t} \mathbf{g}_i(\mathbf{x}, t) \right). \tag{46}$$

In order to transform the above equation, one can use the following formula, which is valid for any nonsingular tensor $\mathbf{g}_i$

$$\mathbf{g}_i^{-1} = \frac{1}{I_3(\mathbf{g}_i)} \left(\frac{\partial I_3(\mathbf{g}_i)}{\partial \mathbf{g}_i} \right)^{T}, \quad I_3(\mathbf{g}_i) = \det(\mathbf{g}_i). \tag{47}$$

After inserting Eq. (47) into Eq. (46) and performing some transformations we obtain

$$\nabla \cdot \mathbf{V}_i = -\frac{1}{I_3(\mathbf{g}_i)} \left(\frac{\delta_i I_3(\mathbf{g}_i)}{\delta t} \right) (i = 1, 2), \tag{48}$$

where we do not imply the summation by repeating subscript. Using the formulae (48) we rewrite the particle balance equations (44) and the mass balance equations (45) as follows

$$\frac{\delta_i}{\delta t} \left(\frac{\eta_i}{I_3(\mathbf{g}_i)} \right) = \frac{\chi_i}{I_3(\mathbf{g}_i)}, \quad \frac{\delta_i}{\delta t} \left(\frac{\rho_i}{I_3(\mathbf{g}_i)} \right) = \frac{\chi_{im}}{I_3(\mathbf{g}_i)} \qquad (i = 1, 2). \tag{49}$$

In what follows we shall primarily discuss the second situation assuming that the number of the fluid particles is not conserved, while the number of the rigid particles is constant. That means that we shall use Eqs. (39)–(40). In this case it is necessary to formulate constitutive equations concerning the functions χ_1, χ_{1m}, χ_{2m}. The functions η_1 and ρ_1 are connected by the relation $m\eta_1 = \rho_1$, where m is the mass of one fluid particle. Thus we have

$$m\chi_1 \equiv -\chi, \quad \chi_{1m} \equiv -\chi, \quad \chi_{2m} \equiv \chi, \quad \chi_2 = 0.$$

The three independent equations from Eqs. (39)–(40) take a form

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \boldsymbol{\nabla} \cdot \mathbf{V}_1 = -\chi, \quad \frac{\delta_2 \eta_2}{\delta t} + \eta_2 \boldsymbol{\nabla} \cdot \mathbf{V}_2 = 0, \quad \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \boldsymbol{\nabla} \cdot \mathbf{V}_2 = \chi$$

or

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \boldsymbol{\nabla} \cdot \mathbf{V}_1 = -\chi, \quad \frac{\delta_2 \eta_2}{\delta t} + \eta_2 \boldsymbol{\nabla} \cdot \mathbf{V}_2 = 0, \quad \frac{\delta_2}{\delta t} \ln \frac{\rho_2}{\eta_2} = \frac{\chi}{\rho_2}. \tag{50}$$

The fraction ρ_2/η_2 has a sense of the variable mass of one particle-fibre.

4 The laws of dynamics

The fundamental laws in the spatial description must be formulated for open systems, i.e. for systems, which interchange with the surrounding medium by mass, momentum, kinetic moment, energy, etc. The momentum of particles for a control volume V is defined as follows

$$\mathbf{K}_1 = \int\limits_{(V)} \left(\rho_1 (\mathbf{x},t)\, \mathbf{V}_1 (\mathbf{x},t) + \rho_2 (\mathbf{x},t)\, \mathbf{V}_2 (\mathbf{x},t)\right) dV(\mathbf{x}) = \int\limits_{(V)} \rho (\mathbf{x},t)\, \mathbf{V}_m (\mathbf{x},t)\, dV(\mathbf{x}). \tag{51}$$

The Euler first law of dynamics is the following statement: *the rate of change of the momentum for an arbitrary physical system is equal to the external force acting on the system plus the external supply of momentum into the system.*

The mathematical form of the first law of dynamics is

$$\frac{d}{dt} \int\limits_{(V)} \rho \mathbf{V}_m dV = \int\limits_{(V)} \rho \mathbf{F} dV + \int\limits_{(S)} \mathbf{T}_{(n)} dS - \int\limits_{(S)} \left[\rho_1 (\mathbf{n} \cdot \mathbf{V}_1)\, \mathbf{V}_1 + \rho_2 (\mathbf{n} \cdot \mathbf{V}_2)\, \mathbf{V}_2\right] dS, \tag{52}$$

where the last integral on the right-hand side is the external supply of momentum into the control volume V.

Using the standard arguments one may introduce the stress tensor $\mathbf{T}$ and derive the Cauchy formulae

$$\mathbf{T}_{(n)} = \mathbf{n} \cdot \mathbf{T} \quad \Rightarrow \quad \int\limits_{(S)} \mathbf{T}_{(n)} dS = \int\limits_{(V)} \boldsymbol{\nabla} \cdot \mathbf{T} dV. \tag{53}$$

Equation (52) takes the form

$$\int\limits_{(V)} \left[(\rho \mathbf{V}_m)^{\cdot} - \rho \mathbf{F} + \boldsymbol{\nabla} \cdot (\rho_1 \mathbf{V}_1 \otimes \mathbf{V}_1 + \rho_2 \mathbf{V}_2 \otimes \mathbf{V}_2) - \boldsymbol{\nabla} \cdot \mathbf{T}\right] dV = \mathbf{0}.$$

Taking into account Eq. (40), the local form of Eq. (52) can be obtained

$$\boldsymbol{\nabla} \cdot \mathbf{T} + \rho \mathbf{F} = \rho_1 \frac{\delta_1 \mathbf{V}_1}{\delta t} + \rho_2 \frac{\delta_2 \mathbf{V}_2}{\delta t} + \chi_{1m} (\mathbf{V}_1 - \mathbf{V}_2). \tag{54}$$

In addition, for the mass density of the external force **F** we may write

$$\rho\,\mathbf{F} = \rho_1\,\mathbf{F}_1 + \rho_2\,\mathbf{F}_2,$$

where the force densities $\mathbf{F}_1$ and $\mathbf{F}_2$ may be of different nature. For example, the rigid particles may be charged.

It is convenient to rewrite Eq. (54) in a form of two equations

$$\nabla\cdot\mathbf{T}' + \rho_1\mathbf{F}_1 + \mathbf{Q} = \rho_1\frac{\delta_1\mathbf{V}_1}{\delta t} + \chi_{1m}\mathbf{V}_1, \quad \nabla\cdot\mathbf{T}'' + \rho_2\mathbf{F}_2 - \mathbf{Q} = \rho_2\frac{\delta_2\mathbf{V}_2}{\delta t} + \chi_{2m}\mathbf{V}_2, \quad (55)$$

where **Q** is the force of interaction between the fluid and the solid-liquid components and

$$\mathbf{T} = \mathbf{T}' + \mathbf{T}'' \quad (56)$$

is postulated. Assuming that

$$\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}_m \equiv \mathbf{V},$$

one can obtain the conventional form of Eq. (54)

$$\nabla\cdot\mathbf{T} + \rho\mathbf{F} = \rho\frac{\delta\mathbf{V}}{\delta t}. \quad (57)$$

The Euler second law of dynamics was established in 1771 and at present is known as the following statement: *the rate of change of the kinetic moment for an arbitrary physical system is equal to the external moment acting on the system plus the external supply of the kinetic moment into the system.*

Let us introduce the kinetic moment of the binary medium

$$\mathbf{K}_2 = \int\limits_{(V)} \rho\mathcal{K}_2\,dV = \int\limits_{(V)} \Big[\underline{\mathbf{x}\times(\rho_1\mathbf{V}_1 + \rho_2\mathbf{V}_2)} + \rho_2\mathbf{J}\cdot\boldsymbol{\omega}\Big]\,dV, \quad (58)$$

where $\rho_2\mathbf{J}$ is the volume density of the inertia tensor of the rigid particles. The underlined term in Eq. (58) is called the moment of momentum.

In the integral form the second law of dynamics can be written as

$$\frac{d}{dt}\int\limits_{(V)} \rho\mathcal{K}_2\,dV = \int\limits_{(V)} (\rho\mathbf{x}\times\mathbf{F} + \rho_2\mathbf{L})\,dV + \int\limits_{(S)} \big(\mathbf{x}\times\mathbf{T}_{(n)} + \mathbf{M}_{(n)}\big)\,dS -$$
$$- \int\limits_{(S)} \mathbf{n}\cdot[\rho_1\mathbf{V}_1\otimes(\mathbf{x}\times\mathbf{V}_1) + \rho_2\mathbf{V}_2\otimes(\mathbf{x}\times\mathbf{V}_2 + \mathbf{J}\cdot\boldsymbol{\omega})]\,dS. \quad (59)$$

In Eqs. (58) and (59) $\mathcal{K}_2$ and **L** denote the densities of the kinetic moment and the external moment, respectively.

Introducing the moment stress tensor **M** and the Cauchy formulae

$$\mathbf{M}_{(n)} = \mathbf{n}\cdot\mathbf{M} \quad \Rightarrow \quad \int\limits_{(S)} \mathbf{M}_{(n)}\,dS = \int\limits_{(V)} \nabla\cdot\mathbf{M}\,dV \quad (60)$$

and taking into account the first law (54), the local form of the second law can be obtained as follows

$$\nabla \cdot \mathbf{M} + \mathbf{T}_{\times} + \rho_2 \mathbf{L} = \rho_2 \frac{\delta_2}{\delta t}(\mathbf{J} \cdot \boldsymbol{\omega}) + \chi_{2m} \mathbf{J} \cdot \boldsymbol{\omega}. \qquad (61)$$

$\mathbf{J}$ is the mass density of the inertia tensor in the actual state. Let $\mathbf{J}_0$ be the inertia tensor in the reference state. Then we have

$$\mathbf{J}(\mathbf{x}, t) = \mathbf{P}(\mathbf{x}, t) \cdot \mathbf{J}_0 \cdot \mathbf{P}^{T}(\mathbf{x}, t). \qquad (62)$$

Let us assume that in the reference state all particles-fibers are transversally isotropic and have the same inertia properties. Thus, we may accept

$$\mathbf{J}_0 = \lambda\, \mathbf{e} \otimes \mathbf{e} + \mu\, (\mathbf{E} - \mathbf{e} \otimes \mathbf{e}), \qquad (63)$$

where the constants λ and μ are the moments of inertia of the rigid particles and the unit vector $\mathbf{e}$ determines the axis of isotropy of the particles in the reference frame.

The reference direction of the vector $\mathbf{e}$ is arbitrary and may be selected to be the same in all points of the reference frame including points which are not occupied by particles-fibres at $t = 0$. Then the reference distribution of the rigid particles can be given by

$$\mathbf{J}(\mathbf{x}_0, 0) = \mathbf{P}_0(\mathbf{x}_0) \cdot \mathbf{J}_0 \cdot \mathbf{P}_0^{T}(\mathbf{x}_0), \quad \mathbf{P}_0(\mathbf{x}_0) \equiv \mathbf{P}(\mathbf{x}_0, 0), \qquad (64)$$

where the rotation tensor $\mathbf{P}_0(\mathbf{x}_0)$ determines the initial orientation of the rigid particles. Let us note that for the considered manufacturing process the distribution of the initial orientations is a random function. Therefore, after solving of a deterministic problem for a given distribution of the tensor $\mathbf{P}_0(\mathbf{x}_0)$ one should solve the problem of the statistical averaging of the results.

If we use Eq. (63) the relation (62) may be rewritten as

$$\mathbf{J}(\mathbf{x}, t) = \mu\, \mathbf{E} + (\lambda - \mu)\, \mathbf{e}' \otimes \mathbf{e}', \quad \mathbf{e}'(\mathbf{x}, t) \equiv \mathbf{P}(\mathbf{x}, t) \cdot \mathbf{e}. \qquad (65)$$

Let us discuss the behavior of the volume density of the inertia tensor $\rho_2(\mathbf{x}, t)\mathbf{J}(\mathbf{x}, t)$ within the flow process. If we take into account the phenomenon of sticking of the fluid particles to the fibers then this tensor is varying. There are two reasons leading to the change of the tensor $\rho_2(\mathbf{x}, t)\mathbf{J}(\mathbf{x}, t)$. The main reason is the variation of the mass density $\rho_2(\mathbf{x}, t)$. However, from the theoretical point of view it is possible to assume that the mass density $\mathbf{J}(\mathbf{x}, t)$ of the inertia tensor is changing too, including both the symmetry properties and the moments of inertia. It seems to be obvious that this second factor is not very important for the considered technological processes. In what follows we shall assume that the volume density of the inertia tensor $\rho_2(\mathbf{x}, t)\mathbf{J}(\mathbf{x}, t)$ may change only due to the changing of the mass density $\rho_2(\mathbf{x}, t)$. In such a case with respect to Eqs. (62) and (30) we have

$$\frac{\delta_2}{\delta t}\mathbf{J}(\mathbf{x}, t) = \boldsymbol{\omega}(\mathbf{x}, t) \times \mathbf{J}(\mathbf{x}, t) - \mathbf{J}(\mathbf{x}, t) \times \boldsymbol{\omega}(\mathbf{x}, t). \qquad (66)$$

Another approach is discussed by Eringen [18, 19], who proposed instead of Eq. (66) the following equation

$$\frac{\delta_2}{\delta t}\mathbf{J}(\mathbf{x}, t) = \boldsymbol{\omega}(\mathbf{x}, t) \times \mathbf{J}(\mathbf{x}, t) - \mathbf{J}(\mathbf{x}, t) \times \boldsymbol{\omega}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t),$$

where the function $\mathbf{f}(\mathbf{x}, t)$ accounts for the sticking of the fluid particles to the rigid particles. This function must be defined by a constitutive equation. In this work we will prefer Eq. (66) which is broadly used in dynamics of rigid body and, in essence, was established by Euler. Let us note that it is possible to take into account the phenomenon of sticking even by use of Eq. (66).

5 Energy balance equation

Each of the fundamental laws introduces a new concept. The first law of dynamics introduces the concept of forces, the second law treats the moments, which are, in general case, not determined through the concept of forces. The third fundamental law in mechanics is the energy balance equation. Within the framework of the continuum mechanics this law plays the most important role, but its formulation is much more difficult in comparison with the first and the second law. The energy balance equation introduces a lot of new concepts. The mostly important of them is the concept of the internal energy. The general formulation of the energy balance equation includes the new concept of the total energy. However, the total energy can be conveniently represented as a sum of the kinetic energy, which has been already defined, and the internal energy, which absorbs all the new concepts contained in the concept of the total energy. One of the principal assumptions within continuum mechanics is the statement that the total energy of a system is an additive function of mass and according to the Radon-Nikodym theorem from the theory of sets, e.g. [23], can be presented as an integral over the mass, where the mass is considered to be a measure. The kinetic energy is, according to its definition, an additive function of mass. Therefore, the additivity of the total energy leads to the additivity of the internal energy. Generally speaking, the additivity of the internal energy is provided only for absolutely continuous systems. However, the known physical world is discrete. Therefore, the assumption about the additivity of the internal energy is a strong restriction. The attempts to relax this restriction are usually based on the concepts of the surface energy or the binding energy. In this work we will follow the traditional assumption about the additivity of the internal energy. In this case the total energy of the binary system can be considered by

$$E = \int\limits_{(V)} \left[\frac{1}{2} (\rho_1 \mathbf{V}_1 \cdot \mathbf{V}_1 + \rho_2 \mathbf{V}_2 \cdot \mathbf{V}_2) + \frac{1}{2} \rho_2 \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} + \rho_1 U_1 + \rho_2 U_2 + \rho U_{12} \right] dV,$$

where U_1 and U_2 are the mass densities of the internal energy of the fluid and the solid-fluid constituents, respectively. U_{12} is the energy of the interaction between the constituents of the binary mixture.

The energy balance equation or the first law of thermodynamics is the following statement: *the rate of change of the total energy of any physical system is equal to the power of external actions on the system plus the rate of the energy supply of the "non-mechanical" nature, usually in the form of heat.* It is difficult to give a general and strict definition of the concept for the energy of the "non-mechanical" nature. Therefore let us restrict ourself by an ambiguous statement that the energy of the "non-mechanical" nature is that part of the energy which is supplied into the system not through the power of external actions.

The energy balance equation may be formulated as follows

$$\frac{dE}{dt} = \int\limits_{(V)} [\rho_1 \mathbf{F}_1 \cdot \mathbf{V}_1 + \rho_2 \mathbf{F}_2 \cdot \mathbf{V}_2 + \rho_2 \mathbf{L} \cdot \boldsymbol{\omega} + \rho\, q]\, dV +$$
$$+ \int\limits_{(S)} \left(\mathbf{T}'_{(n)} \cdot \mathbf{V}_1 + \mathbf{T}''_{(n)} \cdot \mathbf{V}_2 + \mathbf{M}_{(n)} \cdot \boldsymbol{\omega} + h_{(n)}\right) dS - \int\limits_{(S)} \mathbf{n} \cdot \left[\rho_1 \mathbf{V}_1 \left(\frac{1}{2}\mathbf{V}_1 \cdot \mathbf{V}_1 + U_1\right) + \right.$$
$$\left. + \rho_2 \mathbf{V}_2 \left(\frac{1}{2}\mathbf{V}_2 \cdot \mathbf{V}_2 + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} + U_2\right) + \rho \mathbf{V}_m U_{12}\right] dS. \quad (67)$$

In Eq. (67) the decomposition of the total vector of tractions $\mathbf{T}_{(n)}$ is used

$$\mathbf{T}_{(n)} = \mathbf{T}'_{(n)} + \mathbf{T}''_{(n)} = \mathbf{n} \cdot \mathbf{T}' + \mathbf{n} \cdot \mathbf{T}''.$$

q is the rate of production of the energy at a point $\mathbf{x}$ of the reference frame and $h_{(n)}$ is the rate of energy supply through the surface S. The last one can be written using the Stokes rule

$$h_{(n)} = \mathbf{n} \cdot \mathbf{h}, \quad (68)$$

where $\mathbf{h}$ is the vector of the energy flux, which contains all kinds of energy which are not included in the power of external forces and moments. Let us note that in many works the vector $(-\mathbf{h})$ is used instead of the vector $\mathbf{h}$.

Taking into account Eqs. (40), (42), (55) and (61) the energy balance equation (67) may be written down in the local form as

$$\rho_{1m} \frac{\delta_1 U_1}{\delta t} + \rho_{2m} \frac{\delta_2 U_2}{\delta t} + \rho_m \frac{\delta_m U_{12}}{\delta t} =$$
$$= \mathbf{T}'^{T} \cdot\cdot (\nabla \mathbf{V}_1 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{T}''^{T} \cdot\cdot (\nabla \mathbf{V}_2 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{M}^{T} \cdot\cdot \nabla \boldsymbol{\omega} + \mathbf{Q} \cdot (\mathbf{V}_2 - \mathbf{V}_1) +$$
$$+ \nabla \cdot \mathbf{h} + \rho_m q + \chi_{1m} \left(\frac{1}{2}\mathbf{V}_1 \cdot \mathbf{V}_1 - U_1\right) + \chi_{2m} \left(\frac{1}{2}\mathbf{V}_2 \cdot \mathbf{V}_2 + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} - U_2\right). \quad (69)$$

The right-hand side of Eq. (69) contains the power of forces and moments. A part of this power serves for the change of the internal energy. The remaining part of the power partly conserves within the body in the form of heat and partly radiates into the external medium. In order to separate these parts let us introduce the following decompositions

$$\mathbf{T}' = \mathbf{T}'_e + \mathbf{T}'_f, \quad \mathbf{T}'' = \mathbf{T}''_e + \mathbf{T}''_f, \quad \mathbf{M} = \mathbf{M}_e + \mathbf{M}_f, \quad \mathbf{Q} = \mathbf{Q}_e + \mathbf{Q}_f, \quad (70)$$

where the subscript "e" denotes the part which does not depend on the velocities and the subscript "f" denotes the remaining part. In what follows the quantities with subscript "e" will be termed elastic stresses. These elastic stresses always affect the internal energy. The quantities with the subscript "f" may have an influence on the internal energy but only by means of additional parameters like entropy. These parameters will be introduced later.

Substituting Eqs. (70) into Eq. (69) one may obtain

$$\rho_1 \frac{\delta_1 U_1}{\delta t} + \rho_2 \frac{\delta_2 U_2}{\delta t} + \rho \frac{\delta_m U_{12}}{\delta t} =$$
$$= \mathbf{T}_e'^T \cdot\cdot (\nabla \mathbf{V}_1 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{T}_e''^T \cdot\cdot (\nabla \mathbf{V}_2 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{M}_e^T \cdot\cdot \nabla \boldsymbol{\omega} + \mathbf{Q}_e \cdot (\mathbf{V}_2 - \mathbf{V}_1) +$$
$$+ \mathbf{T}_f'^T \cdot\cdot (\nabla \mathbf{V}_1 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{T}_f''^T \cdot\cdot (\nabla \mathbf{V}_2 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{M}_f^T \cdot\cdot \nabla \boldsymbol{\omega} + \mathbf{Q}_f \cdot (\mathbf{V}_2 - \mathbf{V}_1) +$$
$$+ \nabla \cdot \mathbf{h} + \rho_m q + \chi_{1m} \left(\frac{1}{2} \mathbf{V}_1 \cdot \mathbf{V}_1 - U_1 \right) + \chi_{2m} \left(\frac{1}{2} \mathbf{V}_2 \cdot \mathbf{V}_2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} - U_2 \right). \quad (71)$$

Such a form of the energy balance equation is useless. Below we shall transform Eq. (71) in order to obtain the so-called reduced energy balance equation. The idea of such a transformation was discussed in [44].

6 Basic constitutive assumptions

Let us assume that the pressure is moderate, i.e. we will not consider neither the super high nor the super low pressure. That means, that we will exclude the phase transitions of the solid–solid and the liquid–gas types. However, we have to take into account the solid–liquid type phase transitions. In such a case we can assume

$$\mathbf{T}_e' = -p_1(\mathbf{x}, t)\mathbf{E}, \quad \mathbf{T}_e'' = -p_2(\mathbf{x}, t)\mathbf{E}, \quad \mathbf{M}_e = \mathbf{0}, \quad \mathbf{Q}_e = \mathbf{0}. \quad (72)$$

From these assumptions it follows that the rigid particles are not able to form the solid body without strong external loads. Otherwise, we have to take into account the deviatoric part of the stress tensor. Thus, within these assumptions the constituent of particles-fibers behaves like a liquid.

From the intuitive point of view the first three assumptions in Eqs. (72) seem to be quite reasonable. The last assumption in Eqs. (72) is related to the elastic interaction between the fluid and the solid-liquid constituents. Let us note that Eq. (69) includes the quantity $\mathbf{Q} = \mathbf{Q}_e + \mathbf{Q}_f$. The force $\mathbf{Q}_e$ characterizes the elastic interaction and should be determined in such a way that the following equation

$$\mathbf{Q}_e \cdot (\mathbf{V}_1 - \mathbf{V}_2) = \frac{d\mathcal{P}}{dt}, \quad (73)$$

is satisfied, i.e. the elastic force should have a potential. The elastic interaction is present in the nature of the considered phenomenon. Indeed, a moving force field connected with moving particles should be treated using Eq. of the type (73). In the case of a binary medium the right hand side of Eq. (73) should be expressed in terms of two different material derivatives with respect to the velocities $\mathbf{V}_1$ and $\mathbf{V}_2$. This problem requires additional investigation. In this paper we will neglect the elastic interaction assuming $\mathbf{Q}_e = \mathbf{0}$.

With assumptions (72) and taking into account Eqs. (50) the energy balance equation (71) may be rewritten as follows

$$\rho_1 \frac{\delta_1 U_1}{\delta t} + \rho_2 \frac{\delta_2 U_2}{\delta t} + \rho \frac{\delta_m U_{12}}{\delta t} = \frac{p_1}{\rho_1} \frac{\delta_1 \rho_1}{\delta t} + \frac{p_2}{\rho_2} \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \Psi \frac{\delta_2 z}{\delta t} + \nabla \cdot \mathbf{h} + \rho q +$$
$$+ \mathbf{T}_f'^T \cdot\cdot (\nabla \mathbf{V}_1 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{T}_f''^T \cdot\cdot (\nabla \mathbf{V}_2 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{M}_f^T \cdot\cdot \nabla \boldsymbol{\omega} + \mathbf{Q}_f \cdot (\mathbf{V}_2 - \mathbf{V}_1), \quad (74)$$

where

$$z \equiv \ln \frac{\rho_2 \eta_2^0}{\rho_2^0 \eta_2}, \; \Psi \equiv \left(\frac{1}{2} \mathbf{V}_2 \cdot \mathbf{V}_2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} - \frac{p_2}{\rho_2} - U_2 \right) - \left(\frac{1}{2} \mathbf{V}_1 \cdot \mathbf{V}_1 - \frac{p_1}{\rho_1} - U_1 \right) \quad (75)$$

and η_2^0 and ρ_2^0 are the reference density and the mass density of particles-fibers, respectively.

In addition, we have to introduce constitutive assumptions with respect to the forces and the moments of viscous friction. Let us use the conventional notations

$$\mathbf{d} = \frac{1}{2} \left(\nabla \mathbf{V}_1 + \nabla \mathbf{V}_1^T \right), \quad \mathbf{D} = \frac{1}{2} \left(\nabla \mathbf{V}_1 + \nabla \mathbf{V}_1^T - \frac{2}{3} (\nabla \cdot \mathbf{V}_1) \mathbf{E} \right).$$

For the tensor $\mathbf{T}_f'$ let us postulate the following constitutive equation

$$\mathbf{T}_f' = 2 \boldsymbol{\mu} \cdot \cdot \mathbf{D} + \mathbf{t}' \times \mathbf{E}, \quad \mathbf{t}' = \eta_2 \boldsymbol{\mu}_1 \cdot \left(\boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_1 \right). \quad (76)$$

Here the vector $\mathbf{t}'$ characterizes the viscous friction between the solid particles and the fluid. In the first Eq. in (76) the viscosity forth rank tensor $\boldsymbol{\mu}$ must satisfy the following restrictions

$$\mathbf{a} \cdot \cdot \boldsymbol{\mu} \cdot \cdot \mathbf{a} \geq 0, \quad \mathbf{a} \cdot \cdot \boldsymbol{\mu} = \boldsymbol{\mu} \cdot \cdot \mathbf{a}, \quad \mathbf{c} \cdot \cdot \boldsymbol{\mu} = \mathbf{0}, \quad \mathbf{E} \cdot \cdot \boldsymbol{\mu} = \mathbf{0}, \quad \forall\, \mathbf{a}, \mathbf{c} \text{ with } \mathbf{c} = -\mathbf{c}^T, \quad (77)$$

where $\mathbf{a}$ and $\mathbf{c}$ are second rank tensors. Furthermore, if the particle density η_2 vanishes, then the tensor $\boldsymbol{\mu}$ must be isotropic. In the majority of works on suspensions the tensor $\boldsymbol{\mu}$ is supposed to be a transversely isotropic function of $\mathbf{e}'$ and $\mathbf{D}$, where the vector $\mathbf{e}'$ is defined by Eq. (65). Furthermore, the traditional approach assumes that the difference between the suspension and the ordinary fluid lies in the structure of the tensor $\boldsymbol{\mu}$ (see the Introduction to this paper). In this work we do not deny the possibility that the tensor $\boldsymbol{\mu}$ may depend on $\mathbf{e}'$ and $\mathbf{D}$. However, such a dependence is not crucial in our approach. From the physical point of view it seems to be reasonable to assume the tensor $\boldsymbol{\mu}$ to be isotropic.

The viscosity second rank tensor $\boldsymbol{\mu}_1$ in Eq. (76) must satisfy the restrictions

$$\mathbf{a} \cdot \boldsymbol{\mu}_1 \cdot \mathbf{a} \geq 0, \quad \mathbf{a} \cdot \boldsymbol{\mu}_1 = \boldsymbol{\mu}_1 \cdot \mathbf{a}, \quad \forall\, \mathbf{a} \text{ with } |\mathbf{a}| \neq 0, \quad (78)$$

where $\mathbf{a}$ is a vector. Besides, if the particle density η_2 vanishes, then the vector $\mathbf{t}'$ must be zero. We assume that the tensor $\boldsymbol{\mu}_1$ is transversely isotropic

$$\boldsymbol{\mu}_1 = \mu_1^1 \mathbf{e}' \otimes \mathbf{e}' + \mu_1^2 \left(\mathbf{E} - \mathbf{e}' \otimes \mathbf{e}' \right), \quad \mu_1^1 \geq 0, \quad \mu_1^2 \geq 0, \quad (79)$$

where the vector $\mathbf{e}'$ is defined by Eq. (65).

The constitutive equation for the viscous stresses in the solid-liquid constituent may be written as

$$\mathbf{T}_f'' = \mathbf{t}'' \times \mathbf{E}, \quad \mathbf{t}'' = \eta_2 \boldsymbol{\mu}_2 \cdot \left(\boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_2 \right), \quad (80)$$

where the second rank tensor $\boldsymbol{\mu}_2$ has the same form as in Eq. (79). The vector $\mathbf{t}''$ describes the viscous friction between the solid particles. It is clear that the vector $\mathbf{t}''$ must vanish if $\eta_2 = 0$.

The constitutive equation for the viscous moment stress tensor may be accepted in a simple form

$$\mathbf{M}_f = \mathbf{m} \times \mathbf{E}, \quad \mathbf{m} = -\eta_2 \mu_3 \left(\nabla \times \boldsymbol{\omega} \right), \quad \mu_3 \geq 0. \quad (81)$$

Finally, we assume the constitutive equation for the force $\mathbf{Q}_f$ in the following form

$$\mathbf{Q}_f = 2\eta_2 \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_2 - \mathbf{V}_1), \quad \boldsymbol{\mu}_{12} = \mu_{12}^1 \mathbf{e}' \otimes \mathbf{e}' + \mu_{12}^2 (\mathbf{E} - \mathbf{e}' \otimes \mathbf{e}'), \quad \mu_{12}^1 \geq 0,\ \mu_{12}^2 \geq 0. \quad (82)$$

The substitution of Eqs. (76)–(82) into Eq. (74) leads to the following form of the energy balance equation

$$\rho_1 \frac{\delta_1 U_1}{\delta t} + \rho_2 \frac{\delta_2 U_2}{\delta t} + \rho \frac{\delta_m U_{12}}{\delta t} = \frac{p_1}{\rho_1} \frac{\delta_1 \rho_1}{\delta t} + \frac{p_2}{\rho_2} \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \Psi \frac{\delta_2 z}{\delta t} +$$
$$+ \nabla \cdot \mathbf{h} + \rho q + 2\mathbf{D} \cdot \cdot \boldsymbol{\mu} \cdot \cdot \mathbf{D} + 2\eta_2 (\mathbf{V}_2 - \mathbf{V}_1) \cdot \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_2 - \mathbf{V}_1) +$$
$$+ \eta_2 \sum_{i=1}^{2} \left(\boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_i \right) \cdot \boldsymbol{\mu}_i \cdot \left(\boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_i \right) + \eta_2 \mu_3 |\nabla \times \boldsymbol{\omega}|^2. \quad (83)$$

7 The heat conduction equation. The second law of thermodynamics

In order to state the so-called reduced equation of the energy balance we need to define the concepts of temperature, entropy and chemical potential. As a rule, all these concepts are supposed to be well-known [38]. However, in fact there are no satisfactory definitions for them in continuum mechanics. The problem is that it is impossible to prove that the temperature introduced in thermodynamics or in statistical physics coincides with the temperature in continuum mechanics. The same may be said with respect to the entropy and chemical potential. In what follows we will use the approach discussed in [44]. Let us introduce the new variables ϑ_1, ϑ_2, H_1, and H_2 such that

$$\nabla \cdot \mathbf{h} + \rho q + 2\mathbf{D} \cdot \cdot \boldsymbol{\mu} \cdot \cdot \mathbf{D} + 2\eta_2 (\mathbf{V}_2 - \mathbf{V}_1) \cdot \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_2 - \mathbf{V}_1) + \eta_2 \mu_3 |\nabla \times \boldsymbol{\omega}|^2 +$$
$$\eta_2 \sum_{i=1}^{2} \left(\boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_i \right) \cdot \boldsymbol{\mu}_i \cdot \left(\boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_i \right) = \rho_1 \vartheta_1 \frac{\delta_1 H_1}{\delta t} + \rho_2 \vartheta_2 \frac{\delta_2 H_2}{\delta t}, \quad (84)$$

where the parameters ϑ_1 and ϑ_2 will be called temperatures of liquid and solid-liquid species, respectively, and the parameters H_1 and H_2 will be called entropies of the species. The functions ϑ_1 and ϑ_2 are supposed to be measurable by means some experimental procedure. The functions H_1 and H_2 must be defined by means of constitutive equations in such a manner that the temperatures found theoretically coincide with the temperatures found experimentally. From this it follows that the entropy itself has not the meaning of any objective (measurable) quantity. If we change the meaning of temperature, then the meaning of the entropy will be changed as well. Thus we see that Eq. (84) is the true equality rather then additional assumption. In some sense one may say that the right-hand side of Eq. (84) is the notation for the left-hand side of Eq. (84). Equation (84) is termed the heat conduction equation.

Let us rewrite Eq. (84) in an equivalent form using decomposition

$$\mathbf{h} = \mathbf{h}' + \mathbf{h}''.$$

In such a case we have

$$\nabla\cdot\mathbf{h}'+\rho_1 q_1+\mathcal{Q}+2\mathbf{D}\cdot\cdot\boldsymbol{\mu}\cdot\cdot\mathbf{D}+\eta_2\,(\mathbf{V}_2-\mathbf{V}_1)\cdot\boldsymbol{\mu}_{12}\cdot(\mathbf{V}_2-\mathbf{V}_1)+ \\ +\eta_2\left(\boldsymbol{\omega}-\frac{1}{2}\nabla\times\mathbf{V}_1\right)\cdot\boldsymbol{\mu}_1\cdot\left(\boldsymbol{\omega}-\frac{1}{2}\nabla\times\mathbf{V}_1\right)=\rho_1\vartheta_1\frac{\delta_1 H_1}{\delta t},$$

$$\nabla\cdot\mathbf{h}''+\rho_2 q_2-\mathcal{Q}+\eta_2\,(\mathbf{V}_2-\mathbf{V}_1)\cdot\boldsymbol{\mu}_{12}\cdot(\mathbf{V}_2-\mathbf{V}_1)+ \\ +\eta_2\left(\boldsymbol{\omega}-\frac{1}{2}\nabla\times\mathbf{V}_2\right)\cdot\boldsymbol{\mu}_2\cdot\left(\boldsymbol{\omega}-\frac{1}{2}\nabla\times\mathbf{V}_2\right)+\eta_2\mu_3\,|\nabla\times\boldsymbol{\omega}|^2=\rho_2\vartheta_2\frac{\delta_2 H_2}{\delta t}, \quad (85)$$

where the quantity $\mathcal{Q}$ is termed as the heat exchange between the liquid and the solid-liquid constituent. Eq. (84) follows from Eqs. (85). The equivalence of Eq. (84) and Eqs. (85) is determined by the presence of the undefined quantity $\mathcal{Q}$. With the separation of Eq. (84) into two equations Eq. (85) we can state the second law of thermodynamics in a form of two inequalities [42]. The amount of the heat accumulated in each the constituent is determined by the heat exchange $\mathcal{Q}$. For the heat fluxes we apply the Fourier-Stokes law

$$\mathbf{h}'=\kappa_1\nabla\vartheta_1,\quad \mathbf{h}''=\kappa_2\nabla\vartheta_2,\quad \mathcal{Q}=-\kappa\,(\vartheta_1-\vartheta_2),\quad \kappa_1\geq 0,\ \kappa_2\geq 0,\ \kappa\geq 0, \quad (86)$$

where κ_1, κ_2 and κ are the heat conductivities. The latter inequalities do not contradict the second law of thermodynamics which will can be formulated as a set of two inequalities of the Clausius-Duhem type [44]

$$\frac{d}{dt}\int\limits_{(V)}\rho_1 H_1 dV-\int\limits_{(V)}\left[\frac{\rho_1 q_1}{\vartheta_1}+\frac{\mathcal{Q}}{\vartheta_2}\right]dV-\int\limits_{(S)}\mathbf{n}\cdot\left[\frac{\mathbf{h}'}{\vartheta_1}-\rho_1\mathbf{V}_1 H_1\right]dS\geq 0, \quad (87)$$

$$\frac{d}{dt}\int\limits_{(V)}\rho_2 H_2 dV-\int\limits_{(V)}\left[\frac{\rho_2 q_2}{\vartheta_2}-\frac{\mathcal{Q}}{\vartheta_1}\right]dV-\int\limits_{(S)}\mathbf{n}\cdot\left[\frac{\mathbf{h}''}{\vartheta_2}-\rho_2\mathbf{V}_2 H_2\right]dS\geq 0. \quad (88)$$

In the local form the inequalities (87)–(88) may be written as follows

$$\rho_1\frac{\delta_1 H_1}{\delta t}-\frac{1}{\vartheta_1}\left(\nabla\cdot\mathbf{h}'+\rho_1 q_1+\mathcal{Q}\right)+\mathcal{Q}\left(\frac{1}{\vartheta_1}-\frac{1}{\vartheta_2}\right)-\chi H_1+\frac{1}{\vartheta_1^2}\mathbf{h}'\cdot\nabla\vartheta_1\geq 0, \quad (89)$$

$$\rho_2\frac{\delta_2 H_2}{\delta t}-\frac{1}{\vartheta_2}\left(\nabla\cdot\mathbf{h}''+\rho_2 q_2-\mathcal{Q}\right)+\mathcal{Q}\left(\frac{1}{\vartheta_1}-\frac{1}{\vartheta_2}\right)+\chi H_2+\frac{1}{\vartheta_2^2}\mathbf{h}''\cdot\nabla\vartheta_2\geq 0. \quad (90)$$

Making use of Eqs. (85) we obtain

$$2\mathbf{D}\cdot\cdot\boldsymbol{\mu}\cdot\cdot\mathbf{D}+\eta_2\,(\mathbf{V}_2-\mathbf{V}_1)\cdot\boldsymbol{\mu}_{12}\cdot(\mathbf{V}_2-\mathbf{V}_1)+\eta_2\left(\boldsymbol{\omega}-\frac{1}{2}\nabla\times\mathbf{V}_1\right)\cdot\boldsymbol{\mu}_1\cdot\left(\boldsymbol{\omega}-\frac{1}{2}\nabla\times\mathbf{V}_1\right)+ \\ +\mathcal{Q}\left(\frac{1}{\vartheta_1}-\frac{1}{\vartheta_2}\right)-\chi H_1+\frac{1}{\vartheta_1^2}\mathbf{h}'\cdot\nabla\vartheta_1\geq 0, \quad (91)$$

$$\eta_2 (\mathbf{V}_2 - \mathbf{V}_1) \cdot \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_2 - \mathbf{V}_1) + \eta_2 \left(\boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_2 \right) \cdot \boldsymbol{\mu}_2 \cdot \left(\boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_2 \right) +$$
$$+ \eta_2 \mu_3 |\nabla \times \boldsymbol{\omega}|^2 + \mathcal{Q} \left(\frac{1}{\vartheta_1} - \frac{1}{\vartheta_2} \right) + \chi H_2 + \frac{1}{\vartheta_2^2} \mathbf{h}'' \cdot \nabla \vartheta_2 \geq 0. \quad (92)$$

Inequalities (91) and (92) are necessary restrictions which must be valid always and for all processes. If we neglect the sticking of the fluid particles to the solid particles, i.e. assume $\chi = 0$, then the above introduced restrictions for the viscosities (77), (78) and (81) as well as for the heat conductivities (86), are sufficient conditions to satisfy the inequalities (91) and (92). If $\chi \neq 0$, then the inequalities (91) and (92) contain the terms χH_1 and χH_2. From the formal point of view it is not obvious that these inequalities are always satisfied. Nevertheless, we guess that even in this case the inequalities (91) and (92) should be valid without essential restrictions.

8 The reduced energy balance equation. The Cauchy–Green relations

Using Eq. (84) one may rewrite Eq. (83) as follows

$$\rho_1 \frac{\delta_1 U_1}{\delta t} + \rho_2 \frac{\delta_2 U_2}{\delta t} + \rho \frac{\delta_m U_{12}}{\delta t} = \frac{p_1}{\rho_1} \frac{\delta_1 \rho_1}{\delta t} + \frac{p_2}{\rho_2} \frac{\delta_2 \rho_2}{\delta t} + \rho_1 \vartheta_1 \frac{\delta_1 H_1}{\delta t} + \rho_2 \vartheta_2 \frac{\delta_2 H_2}{\delta t} + \rho_2 \Psi \frac{\delta_2 z}{\delta t}. \quad (93)$$

The equation of the energy balance written in the form of Eq. (93) is termed the reduced energy balance equation. From Eq. (93) it is obvious how to define the internal energies U_1, U_2 and U_{12}. The corresponding constitutive equations may be assumed in the simplest form

$$U_1 = U_1(\rho_1, H_1), \quad U_2 = U_2(\rho_2, H_2, z), \quad U_{12} = \text{const.} \quad (94)$$

After the substituting Eq. (94) into Eq. (93) one can derive the Cauchy-Green relations

$$p_1 = \rho_1^2 \frac{\partial U_1}{\partial \rho_1}, \quad p_2 = \rho_2^2 \frac{\partial U_2}{\partial \rho_2}, \quad \vartheta_1 = \frac{\partial U_1}{\partial H_1}, \quad \vartheta_2 = \frac{\partial U_2}{\partial H_2}, \quad \Psi = \frac{\partial U_2}{\partial z}. \quad (95)$$

Instead of internal energies let us introduce the free energies

$$F_1(\rho_1, \vartheta_1) = U_1 - \vartheta_1 H_1, \quad F_2(\rho_2, \vartheta_2, z) = U_2 - \vartheta_2 H_2. \quad (96)$$

The reduced energy balance equation (93) takes now the form

$$\rho_1 \frac{\delta_1 F_1}{\delta t} + \rho_2 \frac{\delta_2 F_2}{\delta t} = \frac{p_1}{\rho_1} \frac{\delta_1 \rho_1}{\delta t} + \frac{p_2}{\rho_2} \frac{\delta_2 \rho_2}{\delta t} - \rho_1 H_1 \frac{\delta_1 \vartheta_1}{\delta t} - \rho_2 H_2 \frac{\delta_2 \vartheta_2}{\delta t} + \rho_2 \Psi \frac{\delta_2 z}{\delta t}. \quad (97)$$

The Cauchy-Green relations (95) can be transformed as follows

$$p_1 = \rho_1^2 \frac{\partial F_1}{\partial \rho_1}, \quad p_2 = \rho_2^2 \frac{\partial F_2}{\partial \rho_2}, \quad H_1 = -\frac{\partial F_1}{\partial \vartheta_1}, \quad H_2 = -\frac{\partial F_2}{\partial \vartheta_2} \quad \Psi = \frac{\partial F_2}{\partial z}. \quad (98)$$

From the last equation in (98) we may conclude that the function Ψ plays the role of the chemical potential. In addition, from the last equations in (95) and (98) it is obvious that the entropy H_2 does not depend on the variable z. In fact, we have

$$\Psi = \frac{\partial F_2}{\partial z} = \frac{\partial U_2}{\partial z} \quad \Rightarrow \quad \frac{\partial \vartheta_2 H_2}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial H_2}{\partial z} = 0.$$

Let us note that the representation of the function Ψ as the derivative of the internal energy is in fact a restriction imposed on the dependence of the internal energy on the variable z. Indeed, accordingly to Eq. (75) and Eq. (95) we have

$$\frac{\partial U_2}{\partial z} = \left(\frac{1}{2}\mathbf{V}_2\cdot\mathbf{V}_2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{J}\cdot\boldsymbol{\omega} - \rho_2\frac{\partial U_2}{\partial \rho_2} - U_2\right) - \left(\frac{1}{2}\mathbf{V}_1\cdot\mathbf{V}_1 - \rho_1\frac{\partial U_1}{\partial \rho_1} - U_1\right). \quad (99)$$

Equation (99) is the partial differential equation for the internal energy. It may be rewritten in the following equivalent form

$$\frac{\partial U_2}{\partial z} + \frac{\partial U_2}{\partial x} = \left(\frac{1}{2}\mathbf{V}_2\cdot\mathbf{V}_2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{J}\cdot\boldsymbol{\omega} - U_2\right) - \left(\frac{1}{2}\mathbf{V}_1\cdot\mathbf{V}_1 - \rho_1\frac{\partial U_1}{\partial \rho_1} - U_1\right), \quad (100)$$

where $x \equiv \ln(\rho_2/\rho_2^0)$. If we introduce the new variables

$$\alpha = \frac{z+x}{2} = \ln\left[\frac{\rho_2}{\rho_2^0}\sqrt{\frac{\eta_2^0}{\eta_2}}\right], \quad \beta = \frac{z-x}{2} = \ln\sqrt{\frac{\eta_2^0}{\eta_2}}, \quad z = \ln\frac{\rho_2\eta_2^0}{\rho_2^0\eta_2}, \quad x = \ln\frac{\rho_2}{\rho_2^0}, \quad (101)$$

then instead of Eq. (100) we obtain

$$\frac{\partial U_2}{\partial \alpha} = \left(\frac{1}{2}\mathbf{V}_2\cdot\mathbf{V}_2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{J}\cdot\boldsymbol{\omega} - U_2\right) - \left(\frac{1}{2}\mathbf{V}_1\cdot\mathbf{V}_1 - \rho_1\frac{\partial U_1}{\partial \rho_1} - U_1\right), \quad (102)$$

where the internal energy U_2 must be considered as a function of α, β, H_2. Furthermore, this function must satisfy the condition (102). Now the Cauchy-Green relations (95) take the form

$$p_1 = \rho_1^2\frac{\partial U_1}{\partial \rho_1}, \quad p_2 = \rho_2\frac{\partial U_2}{\partial \alpha}, \quad \vartheta_1 = \frac{\partial U_1}{\partial H_1}, \quad \vartheta_2 = \frac{\partial U_2}{\partial H_2}, \quad \Psi = \frac{1}{2}\frac{\partial U_2}{\partial \alpha} + \frac{1}{2}\frac{\partial U_2}{\partial \beta}. \quad (103)$$

In terms of the free energy the Cauchy-Green relations may be rewritten as

$$p_1 = \rho_1^2\frac{\partial F_1}{\partial \rho_1}, \quad p_2 = \rho_2\frac{\partial F_2}{\partial \alpha}, \quad H_1 = -\frac{\partial F_1}{\partial \vartheta_1}, \quad H_2 = -\frac{\partial F_2}{\partial \vartheta_2} \quad \Psi = \frac{1}{2}\frac{\partial F_2}{\partial \alpha} + \frac{1}{2}\frac{\partial F_2}{\partial \beta}. \quad (104)$$

Let us recall that the pressure p_2 characterizes the interaction between the fibers. For suspensions under consideration we can assume $p_2 = 0$. In this case we can observe from Eq. (104) that U_2 and F_2 do not depend on α. Consequently, the internal energy U_2 can be found from Eq. (102) as follows

$$U_2 = \left(\frac{1}{2}\mathbf{V}_2\cdot\mathbf{V}_2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{J}\cdot\boldsymbol{\omega}\right) - \left(\frac{1}{2}\mathbf{V}_1\cdot\mathbf{V}_1 - \rho_1\frac{\partial U_1}{\partial \rho_1} - U_1\right). \quad (105)$$

Furthermore, instead of Eqs. (104) we have

$$p_1 = \rho_1^2\frac{\partial F_1}{\partial \rho_1}, \quad p_2 = 0, \quad H_1 = -\frac{\partial F_1}{\partial \vartheta_1}, \quad H_2 = -\frac{\partial F_2}{\partial \vartheta_2} \quad \Psi = \frac{1}{2}\frac{\partial F_2}{\partial \beta}. \quad (106)$$

Here the function Ψ has the exact meaning of the chemical potential. Let us recall that the chemical potential in physics is defined as the derivative of the free energy with respect to the number of the particles in the system under consideration. However, from Eqs. (75) and (102) we see that the chemical potential Ψ is negligibly small. Therefore, the effect of sticking of the fluid particles to the fibres can be ignored. Of course, such a conclusion is valid only within the assumption that the solid particles by itself cannot form a solid body. That means that the distances between the fibres are too large so that the inter-particle forces may be ignored. Thus, we can assume the following representations

$$F_1 = F_1(\rho_1, \vartheta_1), \quad F_2 = F_2(\vartheta_2). \tag{107}$$

Let us introduce a new variable

$$\zeta = \frac{\rho_1^0}{\rho_1} - b, \quad b \equiv \frac{\rho_1^0}{\rho_1^*} \simeq 0.7 \div 0.9; \quad \zeta = 0 \quad \Rightarrow \quad \rho_1 = \rho_1^*, \tag{108}$$

where b is an empirical constant, ρ_1^* is the upper limit of the mass density of the first constituent and ρ_1^0 is the corresponding equilibrium mass density at $p = 0$ and $\vartheta_1 = 0$. Of course, we do not take into account the quantum effects. That means that we consider the case when the temperature is far from the absolute zero. With the variable ζ Eq. (106) can be written as follows

$$p_1 = -\frac{\partial \rho_1^0 F_1}{\partial \zeta}, \quad p_2 = 0, \quad H_1 = -\frac{\partial F_1}{\partial \vartheta_1}, \quad H_2 = -\frac{\partial F_2}{\partial \vartheta_2}, \quad \Psi = 0. \tag{109}$$

The parameter ρ_1^0 will be discussed in more details in the next section. Finally, let us emphasize that from our conclusion it does not follow that the function χ in Eqs. (50) may be ignored.

9 Constitutive equation for the pressure

The constitutive equation for the pressure inside the fluid must be formulated based on known experimental facts. The commonly used approach in models of the suspension flow by the filling is the application of the incompressibility condition, e.g. [4, 3, 7, 10, 15, 39], among others. Furthermore, as far we know, the stage of the solidification was not examined in theoretical works. If we desire to consider the solidification, then we should modify the model of an incompressible fluid. The starting point is the constitutive equation for the pressure proposed in [44]

$$p = p_0 \frac{n}{m-n}\left(\frac{m}{n}\right)^{\frac{n}{m-n}} \left[\left(\frac{1-b}{\zeta}\right)^m - \left(\frac{1-b}{\zeta}\right)^n\right] + \frac{c\vartheta_1}{\zeta^k}, \quad 1 < k < n < m, \quad p_0 > 0, \tag{110}$$

where $p \equiv p_1$ is the pressure in the species 1 and the constant parameters p_0, m, n, k, c, b should be identified experimentally. From the physical point of view it is clear that the constants m, n, k must be odd integers.

Let us discuss the main features of the constitutive equation (110). First of all if the temperature $\vartheta_1 = 0$, then for the pressure $p = 0$ we have $\zeta = 1 - b$ or $\rho_1 = \rho_1^0$, see Fig. 3. Thus, the mass density ρ_1^0 corresponds to the stable equilibrium state of the material in the solid state. The meaning of the constant p_0 in Eq. (110) follows from the expression

$$p_{min} = -p_0, \quad \vartheta_1 = 0.$$

Thus, p_0 is the tensile strength of the material in the solid state at the low temperature. It is important to note that the material under consideration has a finite tensile strength. If the value

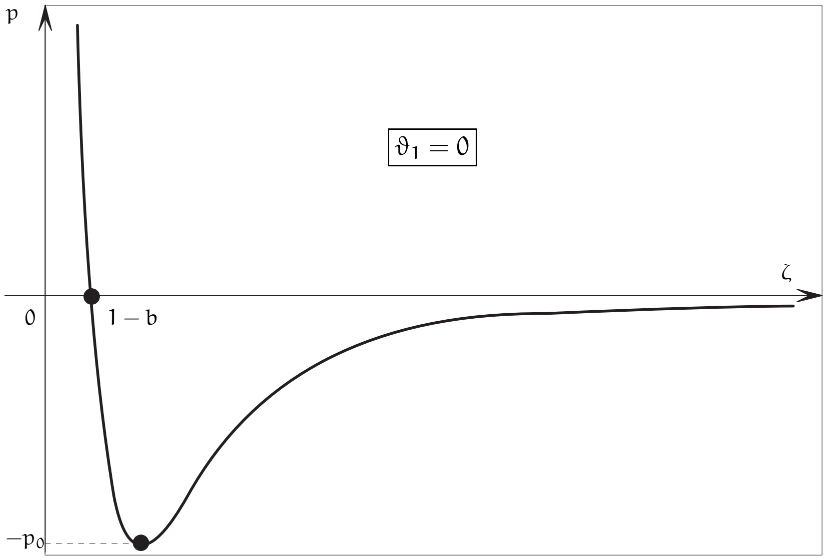

Figure 3. Qualitative variation of the pressure in the solid phase

of the pressure is less than $(-p_0)$, then the material fails. If the value is higher than $(-p_0)$, then the material may exist only in the solid state.

Let us examine the case when $0 < \vartheta_1 < \vartheta_*$, where the temperature ϑ_* will be introduced later. The pressure diagram corresponding to Eq. (110) is shown in Fig. 4. Here we have two equilibrium states with the normalized densities ζ_1 and ζ_2 denoted by the points A and B, respectively, $(1 - b < \zeta_1 < \zeta_2)$. ζ_1 and ζ_2 can be calculated as the roots of the following equation

$$p_0 \frac{n}{m-n}\left(\frac{m}{n}\right)^{\frac{n}{m-n}} \left[\left(\frac{1-b}{\zeta}\right)^m - \left(\frac{1-b}{\zeta}\right)^n\right] + \frac{c\vartheta_1}{\zeta^k} = 0, \quad 0 < \vartheta_1 < \vartheta_*. \qquad (111)$$

The first root ζ_1 corresponds to a stable equilibrium state of the material in the solid phase. The second one ζ_2 corresponds to an unstable equilibrium state of the material. The first zone in Fig. 4 corresponds to the stable solid phase of the material. The pressure within this zone is determined by the constitutive equation (110) and the dependence of the pressure on the density should be verified experimentally. The second zone in Fig. 4 corresponds to the so-called metastable state of the material. Within this zone the material behavior is determined by the equations of motion rather than by the constitutive equation. Let us underline that within this zone there is no static solution or, what is the same, there is an infinite number of static solutions. Furthermore, within this zone we have a mixture of two phase states of the material: the liquid and the solid one. The third zone in Fig. 4 corresponds to the stable liquid phase of the material. In this phase the material can exist only for the pressure lying within the interval $0 < p < p_1$, where p_1 is marked in Fig. 4. Let us assume that the diagram presented in Fig. 4 corresponds to the temperature of polymerization ϑ_p at the pressure p_1. Let the temperature ϑ_p be constant and the pressure p is less than p_1. In such a case we have three equilibrium states denoted by points C, D and E in Fig. 4. Two of these states (points C, E) are stable and the equilibrium state in D is unstable. What state will be realized depends on the initial conditions.

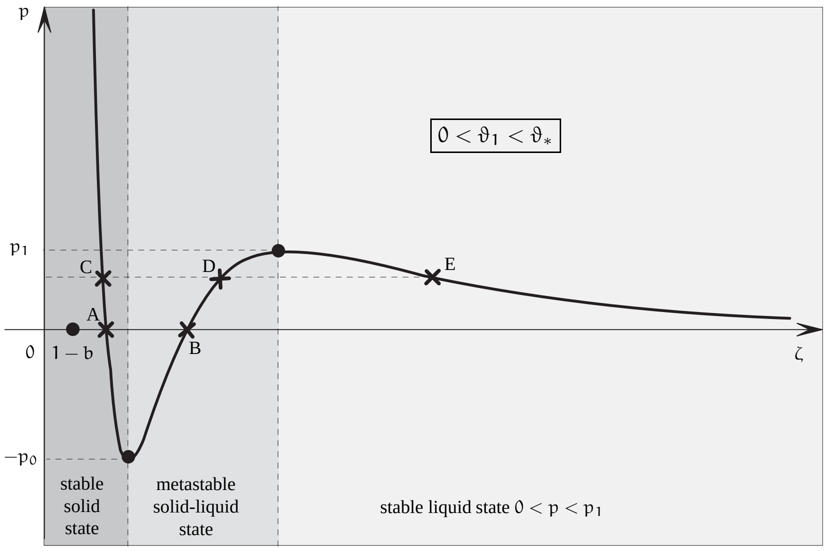

Figure 4. Qualitative variation of the pressure for $0 < \vartheta_1 < \vartheta_*$

If the temperature ϑ_1 increases from 0 up to a value $\vartheta_1 < \vartheta_*$, then the tensile strength p_ϑ of the material decreases to

$$p_\vartheta = -\frac{p_0}{k}\frac{n}{m-n}\left(\frac{m}{n}\right)^{\frac{n}{m-n}}\left[(m-k)\left(\frac{1-b}{\zeta_s(\vartheta_1)}\right)^m - (n-k)\left(\frac{1-b}{\zeta_s(\vartheta_1)}\right)^n\right], \qquad (112)$$

where $\zeta_s(\vartheta_1)$ is the least root of the following equation

$$p_0\frac{n}{m-n}\left(\frac{m}{n}\right)^{\frac{n}{m-n}}\left[m\left(\frac{1-b}{\zeta_s}\right)^m - n\left(\frac{1-b}{\zeta_s}\right)^n\right] + k\frac{c\vartheta_1}{\zeta_s^k} = 0, \quad 0 < \vartheta_1 < \vartheta_*. \qquad (113)$$

The tensile strength p_ϑ of the material should be found experimentally. The polymerization of the suspended fluid is possible only if $\vartheta_1 \leq \vartheta_*$.

Now let us determine the critical temperature ϑ_*. The case $\vartheta = \vartheta_*$ is presented in Fig. 5. Here we have only one equilibrium state at the zero pressure. The material has three different liquid phases. If the pressure p lyes within the range $0 < p < p_1$, then the material has two different liquid states. The first and the third zone in Fig. 5 correspond to the two different stable liquid phases. The second, intermediate zone characterizes an unstable state which corresponds to a mixture of two different liquid phases. If the pressure p is higher than p_1, then we have only one liquid phase. In order to find the density ζ_* corresponding to this state and the critical temperature ϑ_* we have to solve the following system of equations

$$p_0\frac{n}{m-n}\left(\frac{m}{n}\right)^{\frac{n}{m-n}}\left[\left(\frac{1-b}{\zeta_*}\right)^m - \left(\frac{1-b}{\zeta_*}\right)^n\right] + \frac{c\vartheta_*}{\zeta_*^k} = 0,$$

$$p_0\frac{n}{m-n}\left(\frac{m}{n}\right)^{\frac{n}{m-n}}\left[m\left(\frac{1-b}{\zeta_*}\right)^m - n\left(\frac{1-b}{\zeta_*}\right)^n\right] + k\frac{c\vartheta_*}{\zeta_*^k} = 0. \qquad (114)$$

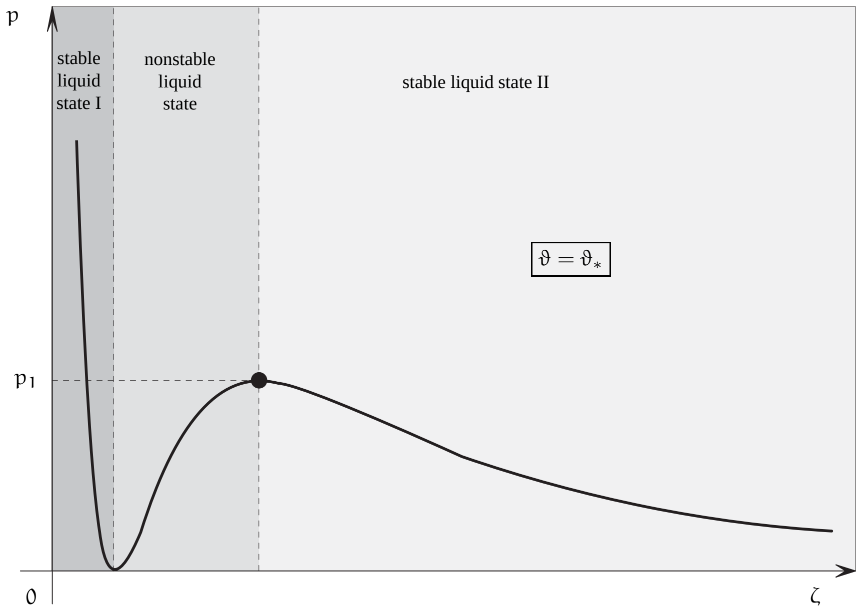

Figure 5. Qualitative variation of the pressure for $\vartheta = \vartheta_*$

The solution of this system may be found as

$$\zeta_* = (1-b)\left(\frac{m-k}{n-k}\right)^{\frac{1}{m-n}}, \quad \frac{c\vartheta_*}{(1-b)^k} = p_0 \frac{n}{m-k}\left(\frac{m}{n}\right)^{\frac{n}{m-n}}\left(\frac{n-k}{m-k}\right)^{\frac{n-k}{m-n}}. \quad (115)$$

Equations (115) can be used in order to find the constants m, n, k since the quantities ζ_* and ϑ_* are experimentally measurable. Let us emphasize that if the temperature ϑ_1 is higher than ϑ_*, then according to the constitutive equation (110) the solidification of the material is impossible. The critical temperature ϑ_* may be termed as the melting temperature.

Figure 6 shows the pressure dependence for the case $\vartheta_* < \vartheta_1 < \vartheta_{**}$. Within this temperature range the material may exist in two liquid phase states. If the temperature ϑ_1 is higher than ϑ_{**}, then the material has only one liquid phase. Let us recall that we do not consider here the gaseous phase of the material. The difference between gas and liquid is that for gas the attractive force decreases more slowly with increasing of ζ. The temperature ϑ_{**} may be found from the equations

$$\frac{dp}{d\zeta} = 0, \quad \frac{d^2p}{d\zeta^2} = 0.$$

By use of Eqs. (109) and (110) we obtain the expression of the free energy

$$\rho_1^0 F_1 = F_0\left[-\frac{\zeta}{m-1}\left(\frac{1-b}{\zeta}\right)^m + \frac{\zeta}{n-1}\left(\frac{1-b}{\zeta}\right)^n\right] - \frac{1}{k-1}\frac{c\vartheta_1}{\zeta^{k-1}} + \psi(\vartheta_1), \quad (116)$$

where the function $\psi(\vartheta_1)$ must be specified for the given material. One may find the function

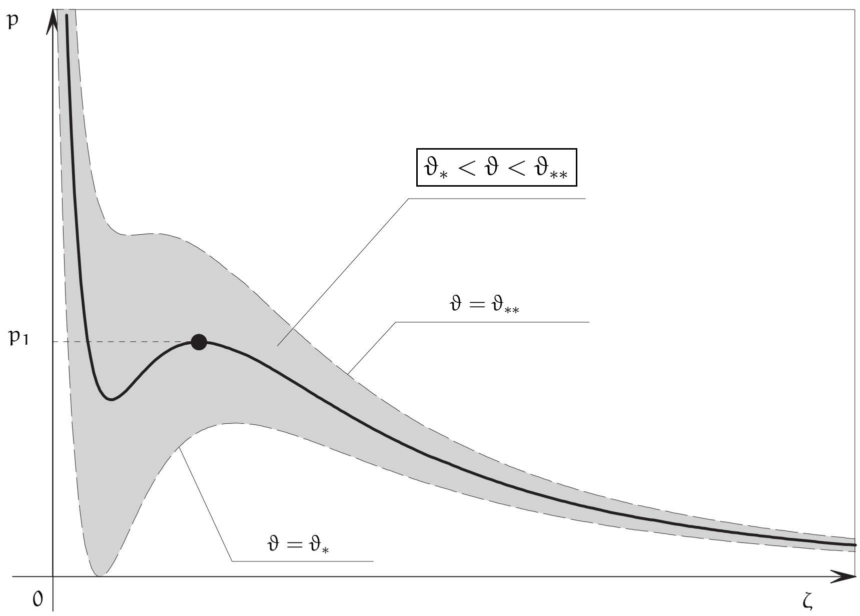

Figure 6. Qualitative variation of the pressure for $\vartheta_* < \vartheta_1 < \vartheta_{**}$

$\psi(\vartheta_1)$ from the following equation

$$-\frac{\partial\psi(\vartheta_1)}{\partial\vartheta_1} = c_\varepsilon \ln \frac{\vartheta_1}{\vartheta_0},$$

where c_ε is the heat capacity at constant strains.

10 The final system of equations

The basic unknowns of the considered problem are ρ_1 and η_2, ρ_2. For these functions we have the set of equations (50)

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \nabla \cdot \mathbf{V}_1 = -\chi, \quad \frac{\delta_2 \eta_2}{\delta t} + \eta_2 \nabla \cdot \mathbf{V}_2 = 0, \quad \frac{\delta_2}{\delta t} \ln \frac{\rho_2}{\eta_2} = \frac{\chi}{\rho_2}, \tag{117}$$

where the function χ should be specified. In the simplest case it is possible to assume that $\chi = 0$.

The equations of motion of the liquid constituent are given by the first equation from Eqs. (55). Taking into account the constitutive equation (76) we obtain

$$-\nabla p_1 + 2\nabla \cdot (\boldsymbol{\mu} \cdot\cdot \mathbf{D}) + \nabla \times \left[\eta_2 \boldsymbol{\mu}_1 \cdot \left(\boldsymbol{\omega} - \frac{1}{2}\nabla \times \mathbf{V}_1\right)\right] + \\ + \eta_2 \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_2 - \mathbf{V}_1) + \rho_1 \mathbf{F}_1 = \rho_1 \frac{\delta_1 \mathbf{V}_1}{\delta t} - \chi \mathbf{V}_1. \tag{118}$$

The equations of the translation motion of the solid-liquid constituent take the form

$$-\nabla p_2+\nabla\times\left[\eta_2\boldsymbol{\mu}_2\cdot\left(\boldsymbol{\omega}-\frac{1}{2}\nabla\times\mathbf{V}_2\right)\right]+\eta_2\boldsymbol{\mu}_{12}\cdot(\mathbf{V}_1-\mathbf{V}_2)+\rho_2\mathbf{F}_2=\rho_2\frac{\delta_2\mathbf{V}_2}{\delta t}+\chi\mathbf{V}_2, \quad (119)$$

where the partial pressure p_2 may be assumed to be zero, or may be defined by a constitutive equation like Eq. (110).

Equation (61) for the spinor motion of fibres can be written as follows

$$-\eta_2\mu_3\nabla\times(\nabla\times\boldsymbol{\omega})-2\eta_2\boldsymbol{\mu}_2\cdot\left(\boldsymbol{\omega}-\frac{1}{2}\nabla\times\mathbf{V}_2\right)-2\eta_2\boldsymbol{\mu}_1\cdot\left(\boldsymbol{\omega}-\frac{1}{2}\nabla\times\mathbf{V}_1\right)+\rho_2\mathbf{L}=$$
$$=\rho_2\frac{\delta_2}{\delta t}(\mathbf{J}\cdot\boldsymbol{\omega})+\chi(\mathbf{J}\cdot\boldsymbol{\omega}). \quad (120)$$

To the above equations (117)–(120) we have to add the heat conduction equations (85) together with constitutive equations (86) in which it is possible to assume that $\kappa_2=0$.

In order to obtain the final statement of the basic equations we have to determine the volume forces $\rho_1\mathbf{F}_1$, $\rho_2\mathbf{F}_2$ and the volume moment $\rho_2\mathbf{L}$. Partly they are determined by the external fields, usually it is the gravity field, which does not create the external moment $\rho_2\mathbf{L}$. Besides, the volume forces and moments may arise due to the boundary walls. Let us suppose, that the boundary of the mold cavity V is bounded by the surface S. Let this surface consists of two parts $S=S_0\cup S_1$, where S_0 is the rigid wall and S_1 is an inlet of the cavity through which the mixture flows into the cavity V. Let $\mathbf{n}$ be a unit normal vector to S, directed towards the domain V. Let s be a distance along the normal $\mathbf{n}$. Let us assume that the influence of the wall may be described in terms of the external force field which must be considered by means of the volume forces. Let us specify the volume forces as follows

$$\mathbf{L}=\mathbf{0}, \quad \mathbf{F}_1=\mathbf{F}_2=\mathbf{g}+F_0\left[\left(\frac{s}{l}\right)^{-p}-\left(\frac{s}{l}\right)^{-q}\right]\mathbf{n}, \quad p>q>0, \quad s\geq 0, \quad (121)$$

where $l>0$ is a very small constant having the dimension of length. In general, the wall creates the moment acting on the rigid particles. But we shall neglect by this moment. The mixture in fluid state is inserted into the cavity through the inlet S_1 and occupies some domain V_* which is changing in time. The boundary of V_* is the surface S_* that consists of three parts $S_*=S_1\cup S_0^*\cup S_f$, where S_0^* is those part of S_0 which is in contact with the mixture and S_f is the free surface of the mixture. For all these surfaces we have to state the boundary conditions which may be prescribed in the conventional form for the velocities and the pressure.

11 Discussion

The aim of this paper was to derive the governing equations describing the flow of the fiber suspension within the framework of the micro-polar model of a binary medium. In a forthcoming paper the numerical solution of the proposed equation as well as some illustrating examples will be presented.

As a discussion let us compare our approach with the existing theories. For this purpose we will consider simplifying assumptions as it made within the existing theories. Let us proceed by several steps.

The first step is the simplest one. Let us assume that

$$\vartheta_1 = \vartheta_2 = \text{const}, \quad \chi = 0. \tag{122}$$

In this case from Eqs. (117) follows

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \nabla \cdot \mathbf{V}_1 = 0, \quad \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \nabla \cdot \mathbf{V}_2 = 0 \quad \Rightarrow \quad \frac{d\rho}{dt} + \nabla \cdot (\rho \mathbf{V}_m) = 0, \tag{123}$$

where $\rho = \rho_1 + \rho_2$ and $\rho \mathbf{V}_m = \rho_1 \mathbf{V}_1 + \rho_2 \mathbf{V}_2$.

The next assumption of the conventional theory is that there is no sliding between the rigid particles and the fluid. From the physical point of view that means that the norms of the viscous friction tensors strive to infinity

$$||\boldsymbol{\mu}_1|| \to \infty, \quad ||\boldsymbol{\mu}_2|| \to \infty, \quad ||\boldsymbol{\mu}_{12}|| \to \infty. \tag{124}$$

Since the norms of vectors $\mathbf{t}'$, $\mathbf{t}''$ and $\mathbf{Q}$ must be limited then from the constitutive equations (76), (80) and (82) we obtain the following restrictions

$$\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}_m \equiv \mathbf{V}, \quad 2\boldsymbol{\omega} = \nabla \times \mathbf{V}, \tag{125}$$

which are usually assumed except, may be, the last condition. However, this condition must be valid too. In this case instead of equations of motion (118)–(120) we shall get

$$\begin{aligned} -\nabla p_1 + 2\nabla \cdot (\boldsymbol{\mu} \cdot \cdot \mathbf{D}) + \nabla \times \mathbf{t}' + \mathbf{Q} + \rho_1 \mathbf{F}_1 &= \rho_1 \frac{\delta \mathbf{V}}{\delta t}, \\ -\nabla p_2 + \nabla \times \mathbf{t}'' - \mathbf{Q} + \rho_2 \mathbf{F}_2 &= \rho_2 \frac{\delta \mathbf{V}}{\delta t}, \quad p_2 \simeq 0, \\ -\eta_2 \mu_3 \nabla \times (\nabla \times \boldsymbol{\omega}) - 2\mathbf{t} + \rho_2 \mathbf{L} &= \rho_2 \frac{\delta}{\delta t} (\mathbf{J} \cdot \boldsymbol{\omega}), \end{aligned} \tag{126}$$

where the vectors $\mathbf{t}'$, $\mathbf{t}''$, $\mathbf{Q}$ are not defined by constitutive equations any more. We may rewrite this system as a set of two equations

$$\begin{aligned} -\nabla p_1 + 2\nabla \cdot (\boldsymbol{\mu} \cdot \cdot \mathbf{D}) + \nabla \times \mathbf{t} + \rho \mathbf{F} = \rho \frac{\delta \mathbf{V}}{\delta t}, \quad 2\boldsymbol{\omega} = \nabla \times \mathbf{V}, \\ -\eta_2 \mu_3 \nabla \times (\nabla \times \boldsymbol{\omega}) - 2\mathbf{t} + \rho_2 \mathbf{L} = \rho_2 \frac{\delta}{\delta t} (\mathbf{J} \cdot \boldsymbol{\omega}). \end{aligned} \tag{127}$$

The latter equation allows to express the vector $\mathbf{t}$ in terms of the velocity $\mathbf{V}$ and its derivatives. Generally speaking, further simplifications are not possible. However, Eqs. (127) are more complicated than those used in the conventional theory.

To proceed with one more step forward to the conventional theory we have to assume that

$$\mu_3 = 0, \quad \mathbf{L} = \mathbf{0}, \quad \mathbf{J} = \mathbf{0} \quad \Rightarrow \quad \mathbf{t} = \mathbf{0}. \tag{128}$$

The first from these restrictions is the hypothesis of the dilute mixture. The third one is the hypothesis of the inertialess particles. Now we have the following equations

$$-\nabla p_1 + 2\nabla \cdot (\boldsymbol{\mu} \cdot \cdot \mathbf{D}) + \rho \mathbf{F} = \rho \frac{\delta \mathbf{V}}{\delta t}, \quad \boldsymbol{\omega} = \frac{1}{2} \nabla \times \mathbf{V}, \quad (\nabla \cdot \boldsymbol{\omega} = 0). \tag{129}$$

If we introduce a unit vector $\mathbf{m}(t)$ connected to the fiber symmetry axis, then we obtain

$$\frac{\delta \mathbf{m}(t)}{\delta t} = \frac{1}{2} (\nabla \times \mathbf{V}) \times \mathbf{m}(t) = \frac{1}{2} (\mathbf{m} \cdot \nabla \mathbf{V} - \nabla \mathbf{V} \cdot \mathbf{m}) . \tag{130}$$

The last, but not least restriction is that of incompressibility

$$\nabla \cdot \mathbf{V} = 0, \quad \frac{\delta \rho}{\delta t} = 0. \tag{131}$$

This condition means that the pressure is not defined by the constitutive equation anymore. It must be found from the equations of motion. Thus, the phase transitions in the mixture must be ignored. Let us note, that the condition $\rho = \text{const}$ does not follow from Eq. (131).

Thus, after all assumptions we have obtained the set of equations (129)–(131). We have to add to these equations the statement that the viscous tensor $\boldsymbol{\mu}$ is a transversely isotropic function of the unit vector **m** and the tensor **D**. We do not discuss this question since the solution of this problem is well-known. It is easy to see that the set of equations (129)–(131) does not correspond exactly to that in the conventional theory (see the Introduction to this paper). The only difference is that Eq. (130) does not coincide with the Eq. (13). In order to see this difference more clearly let us rewrite Eq. (13) in the equivalent form

$$\dot{\mathbf{m}} = \left(\frac{1}{2} \nabla \times \mathbf{V} + \frac{c^2 - a^2}{c^2 + a^2} \mathbf{m} \times \mathbf{d} \cdot \mathbf{m} \right) \times \mathbf{m}, \quad \mathbf{d} = \frac{1}{2} \left(\nabla \mathbf{V} + \nabla \mathbf{V}^T \right) . \tag{132}$$

What equation (Eq. (130) or Eq. (132)) is more right? We do not know the exact answer. On one hand, Eq. (132) seems to be more preferable since it contains the parameters of the particle. On the other hand, this fact seems to be strange if we assume that there is no sliding between the solid particle and the fluid. Consequently the rotations of the solid particle and the fluid must be the same. That means that the shape of a particle should be not important. It is true for Eq. (130) but it is not true for Eq. (132). Let us note that Eq. (130) may be derived by many ways.

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Symmetries and Orthogonal Invariants in Oriented Space*

Abstract

The theory of the tensor symmetry is modified in order to include into consideration the Non-Euclidean tensors. The polar (Euclidean) and axial (Non-Euclidean) tensors are discussed. A new definition of the tensor invariants is given. This definition coincides with the conventional one only for the Euclidean tensors. It is shown that any invariant is the solution of some partial differential equation of the first order. The number of the independent solutions of this equation determines the minimal number of the invariants which are necessary in order to fix a system of tensors as a rigid whole. This result was not found in the known publications. The examples of some systems of tensors are discussed in order to give the comparison with known results.

1 Introduction

Symmetries and orthogonal invariants are important theoretical tools for many fields of mechanics. Therefore these tools must be applicable to all objects widely used in mechanics. Unfortunately this is not so. The application of the classical theory of symmetry leads to the meaningless results in shell theory and not only in shell theory. The main reason of this is that in many cases we are forced to work in multi-oriented spaces. The classical theory of symmetry and invariants is well defined in non-oriented vector space only. In order to define the cross product of vectors we have to introduce the oriented vector space. There are two different types of tensors acting in oriented space. These tensors are known as polar and axial ones. In oriented space the classical theory of symmetry is well defined for polar tensors. There exist many formally equivalent ways for introduction of the space orientation. In this paper we introduce a definition of the space orientation in such a way that the physical sense of this concept is quiet clear. Besides we restrict ourselves by consideration of the oriented space. In general case this is not enough. For example, in shell theory it is necessary to use multi-oriented space. Briefly speaking in shell theory 3D-space E_3 must be represented as a direct sum of 2D-space E_2 and 1D-space E_1: $\mathsf{E}_3 = \mathsf{E}_2 \oplus \mathsf{E}_1$. If we orientate each of these spaces, then we obtain three oriented spaces $\mathsf{E}_3^{\mathsf{O}}$, $\mathsf{E}_2^{\mathsf{O}}$ and $\mathsf{E}_1^{\mathsf{O}}$. Suppose that there is a relation $\mathsf{E}_3^{\mathsf{O}} = \mathsf{E}_2^{\mathsf{O}} \oplus \mathsf{E}_1^{\mathsf{O}}$. In such a case only

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two orientations are independent and we have 2-oriented space. In this 2-oriented space the four different types of tensors may be defined. The classical theory of symmetry is correct for one of these types of tensors called polar tensors. The modified theory of symmetry for other types of tensors may be found in the paper [1].

At the moment oriented space is the most important and popular vector space in mechanics. By this reason the paper deals with the symmetries and orthogonal invariants in oriented space. Necessary generalizations may be obtained without any problems.

1.1 Classical theory

The classical theory of the tensor symmetry and the tensor invariants was born due to O. Cauchy and is extensively developing up to now. The books [2, 3, 4] contain the conventional statements of the problem. The modern applications of symmetries and invariants to mechanics may be found in the book [5]. Recall that almost all modern results of the invariant theory are obtained for the polynomial invariants [6]. The sufficiently complete list of the modern papers on the subject may be found in the paper [7].

In this subsection we reproduce the basic definitions of the classical theory in order to avoid the possible misunderstandings. In the sequel the direct tensor notation [8] is used. In some works [6] the term "direct notation" has another meaning: a notation $\mathbf{a}$ is assigned to the triple of vector coordinates.

In the sequel the next notation will be used

$$\underbrace{f,\ g, \ldots, h}_{\text{scalars}};\quad \underbrace{\mathbf{a} \equiv a^i \mathbf{g}_i,\ \mathbf{b}, \ldots, \mathbf{c}}_{\text{vectors}};\quad \underbrace{\mathbf{A} \equiv A^{ij} \mathbf{g}_i \otimes \mathbf{g}_j,\ \mathbf{B}, \ldots, \mathbf{C}}_{\text{2-rank tensors}};\quad \ldots,$$

where vectors $\mathbf{g}_i$ consist arbitrary basis in the reference system.

The set of second-rank tensors $\mathbf{Q}$ such that

$$\mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{E}, \quad \det \mathbf{Q} = \pm 1$$

is called orthogonal group, which contains infinitely many different elements but any of them may be generated by two orthogonal tensors. First of them is the tensor of a mirror reflection from the plane with unit normal $\mathbf{n}$. This tensor is determined by the expression

$$\mathbf{Q} = \mathbf{E} - 2\,\mathbf{n} \otimes \mathbf{n}, \quad \mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{E}, \quad \det \mathbf{Q} = -1. \tag{1}$$

The second tensor is the tensor of turn (rotation). With the help of Euler's theorem this tensor may be represented in the following form

$$\mathbf{Q}(\varphi \mathbf{m}) \equiv (1 - \cos\varphi)\mathbf{m} \otimes \mathbf{m} + \cos\varphi \mathbf{E} + \sin\varphi\, \mathbf{m} \times \mathbf{E}, \quad \det \mathbf{Q} = +1, \tag{2}$$

where the unit vector $\mathbf{m}$ determines a strait line called the turn axis, an angle φ is called the turn angle. The action of the tensor (2) on a vector $\mathbf{a}$ is the turn of $\mathbf{a}$ around the vector $\mathbf{m}$ by the angle φ. Any orthogonal tensor may be represented as the composition of the tensors (1) and (2).

In classical theory the orthogonal transformation of the n-rank tensor $\mathbf{D}$ is defined by the formula

$$\mathbf{D}' \equiv \mathbf{Q}^n \odot \mathbf{D} \equiv \mathbf{Q}^n \odot \left(D^{i_1 \ldots i_n} \mathbf{g}_{i_1} \otimes \ldots \otimes \mathbf{g}_{i_n}\right) \equiv D^{i_1 \ldots i_n} \mathbf{Q} \cdot \mathbf{g}_{i_1} \otimes \ldots \otimes \mathbf{Q} \cdot \mathbf{g}_{i_n} \tag{3}$$

For example, for scalars, vectors and 2-rank tensors we have

$$f' \equiv f, \quad \mathbf{a}' \equiv \mathbf{Q} \cdot \mathbf{a}, \quad \mathbf{A}' \equiv \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^T.$$

Classical definition of the symmetry group*: the sets of orthogonal solutions of the equations*

$$\mathbf{a}' \equiv \mathbf{Q} \cdot \mathbf{a} = \mathbf{a}, \quad \mathbf{A}' \equiv \mathbf{Q} \cdot \mathbf{A} \cdot \mathbf{Q}^T = \mathbf{A}, \quad \mathbf{D}' \equiv \mathbf{Q}^n \odot \mathbf{D} = \mathbf{D} \tag{4}$$

are called the groups of symmetry (SG) of the vector $\mathbf{a}$*, 2-rank tensor* $\mathbf{A}$ *and n-rank tensor* $\mathbf{D}$ *correspondingly, where the vector* $\mathbf{a}$*, 2-rank tensor* $\mathbf{A}$ *and n-rank tensor* $\mathbf{D}$ *are given and orthogonal tensors* $\mathbf{Q}$ *must be found.*

Definition. *The n-rank tensor* $\mathbf{D}$ *is called isotropic if its group of symmetry contains all orthogonal tensors.*

There are two basic problems in the theory of symmetry:

Direct problem*: to find SG for the given system of tensors*.

Inverse problem*: to find the structure of a tensor of some order with given elements of symmetry*.

In non-oriented vector space the definition (4) leads to the correct results both from the mathematical and from physical points of view. However in oriented vector space with cross product of vectors this definition generates some paradoxical results from physical point of view.

Physical paradoxes.

1. Let $\mathbf{V}$ be a vector of translation velocity and $\boldsymbol{\omega}$ be a vector of the angular velocity such that $\mathbf{V} \times \boldsymbol{\omega} = 0$. Accordingly to the definition (4) these vectors have the same groups of symmetry. This is nonsense from physical point of view.

2. Tensor $\mathbf{E} \times \mathbf{E}$ is not isotropic. From physical point of view this result seems to be doubtful.

Definition*. A scalar-valued tensor function* $\psi(f, \mathbf{a}, \mathbf{A})$ *is said to be the orthogonal invariant if the equation*

$$\psi(f', \mathbf{a}', \mathbf{A}') = \psi(f, \mathbf{a}, \mathbf{A}) \tag{5}$$

holds for all orthogonal tensors $\mathbf{Q}$.

Theorem *(Gilbert). For any finite system of tensors there exist a finite basis of invariants, that means the finite system of the functionally independent scalar invariants such that all other invariants can be expressed in terms of basis invariants.*

The central problem in classical invariant theory: *for a given set of tensors and a given transformation group, determine a set of invariants from which all other invariants can be generated.*

Physical paradox: *a mixed product of three vectors* $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ *is not invariant with respect to orthogonal group.*

It is easy to give a lot of other examples in which the classical theory leads to the physical mistakes. All these examples are connected with axial objects in oriented system of reference [9]. A lot of important discussions on the subject may be found in [10] – [22]. Some of them will be considered below.

1.2 A modified statement of the problem

The main purpose of the paper is to modify the classical theory in order to avoid the contradictions between mathematics and physics.

For this end we have to slightly change the definitions of the invariants and groups of symmetry. For example, the problem of invariants may be reformulated by the next manner. Let there be given two sets of tensors

$$\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_m, \quad \mathbf{A}_1, \mathbf{A}_2, \dots \mathbf{A}_n \tag{6}$$

and

$$\mathbf{b}_1, \mathbf{b}_2, \dots \mathbf{b}_m, \quad \mathbf{B}_1, \mathbf{B}_2, \dots \mathbf{B}_n. \tag{7}$$

The invariant problem. *To find the minimal collections of the invariants for the system (6) and (7) whose coinciding is the guarantee of existing of the proper orthogonal tensor* $\mathbf{P}$: det $\mathbf{P} = 1$ *such that equalities*

$$\mathbf{b}_1 = \mathbf{P} \cdot \mathbf{a}_1, \quad \mathbf{b}_2 = \mathbf{P} \cdot \mathbf{a}_2, \dots, \mathbf{b}_m = \mathbf{P} \cdot \mathbf{a}_m,$$

$$\mathbf{B}_1 = \mathbf{P} \cdot \mathbf{A}_1 \cdot \mathbf{P}^T, \quad \dots, \quad \mathbf{B}_n = \mathbf{P} \cdot \mathbf{A}_n \cdot \mathbf{P}^T \tag{8}$$

holds. This statement will be called ***I-problem*** in what follows.

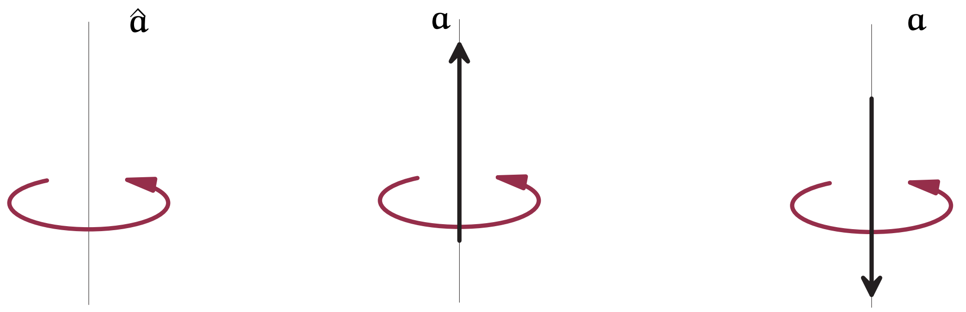
In other words, if the basis invariants for the system (6) and (7) coincide, then the system (7) may be obtained from the system (6) by the rigid rotation.

In the classical statement of the problem the tensor $\mathbf{P}$ in (8) may be orthogonal one rather then proper orthogonal tensor. For physical applications the tensor $\mathbf{P}$ must be the proper orthogonal tensor. This fact will be shown in what follows.

2 Orientation of Reference System. Polar and Axial Objects

The necessity of orientation of reference system arises due to our desire to take into account the moment interaction in mechanics. In the nature there are two principally different kinds of motion: the translation motion and the spinor (rotational) motion. Under translational motion a body is changing the position in the space. Under spinor motion a body is changing an orientation in the space without changing of position. The changing of translational motion is determined by forces. The changing of spinor motion is determined by moments. Note that in general moments can not be reduced to the concept of the force moment. In order to describe the spinor movements and the moment interactions we must orient the reference system and to introduce some new objects called axial objects in addition to the conventional objects called polar. There are many different but mathematically almost equivalent ways to introduce the space orientation. We prefer a way with clear physical sense. The physical (and mathematical) image of the spinor movement is given by so-called spin-vector whose introduction does not require the space orientation. Let there be given some system of reference (SR). Polar vector is represented in SR as an arrow. In addition to polar vector let us introduce a new object called spin-vector. For this it is necessary to take a strait line in SR called axis of a spin-vector. After that a circular arrow around the axis of a spin-vector must be drown in the plain orthogonal to the axis. Now we have a visual image of the spin-vector — see Figure 1a.

The length of the circular arrow is called a modulus of the spin-vector. A direction of a circular arrow shows the direction of a rotation. Spin-vectors describe characteristics of spinor movements. They are convenient for an intuition. However for the formal calculations it is much better to use so-called axial vectors. An axial vector can be obtained from a spin-vector with the



a). Object $\hat{\mathbf{a}}$ is called a spin vector

b). Strait arrow is the axial vector $\mathbf{a}$ corresponding to $\hat{\mathbf{a}}$ in the right-oriented SR

c). Strait arrow is the axial vector $\mathbf{a}$ corresponding to $\hat{\mathbf{a}}$ in the left-oriented SR

Figure 1: Oriented System of Reference

help of special rule called an orientation of the reference system. An axial vector $\mathbf{a}$ is associated with the spin-vector $\hat{\mathbf{a}}$ by means of the following rule:

1) $\mathbf{a}$ is placed on the axis of spin-vector $\hat{\mathbf{a}}$, 2) modulus of $\mathbf{a}$ is equal to the modulus of $\hat{\mathbf{a}}$, 3) the vector $\mathbf{a}$ is directed as shown at the Fig.1b (in such a case we have the right-oriented SR) **or** *as shown at the Fig.1c (in such a case we have the left-oriented SR).*

The concept of an axial vector introduced above is the exact expression of a physical idea about angular velocity, moment and so on. The introduction of axial vector does not require any system of coordinates. For example, the coordinate free introduction of the cross product of vectors with the using of spin-vector may be found in [9]. Let us consider two different coordinate systems in SR. Let $\mathbf{g}_i$ and $\mathbf{g}_{i'}$ be the local bases of these coordinate systems. If

$$\mathbf{g}_{i'} = h^m_{i'}\mathbf{g}_m, \quad \mathbf{g}^{i'} = h^{i'}_m\mathbf{g}^m, \quad h^{i'}_m h^m_{k'} = \delta^i_k, \quad h^{i'}_m h^n_{i'} = \delta^n_m,$$

then we have

$$\mathbf{a} = a^i\mathbf{g}_i = a^{i'}\mathbf{g}_{i'}, \quad a^{i'} = h^{i'}_m a^m. \tag{9}$$

This transformation of the vector components is valid both for polar vectors and for axial vectors. Here there is a contradiction with the conventional determinations for axial vectors whose coordinates are transformed accordingly to the rule (see, for example, [6])

$$a^{i'} = \det(h^{k'}_n)h^{i'}_m a^m,$$

where matrix $h^{k'}_n$ is supposed to be orthogonal.

From the pure mathematical point of view this definition is possible. However from physical point of view only the definition (9) must be used. Indeed the simplest example of axial vector is given by the cross product of two polar vectors

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} \quad \Rightarrow \quad c_p = a^m b^n (\mathbf{g}_m \times \mathbf{g}_n) \cdot \mathbf{g}_p.$$

In other system of coordinates we have

$$c_{p'} = a^{m'} b^{n'} (\mathbf{g}_{m'} \times \mathbf{g}_{n'}) \cdot \mathbf{g}_{p'} = a^r h^{m'}_r b^s h^{n'}_s (\mathbf{g}_{m'} \times \mathbf{g}_{n'}) \cdot h^t_{p'} \mathbf{g}_t =$$

$$= h^{t}_{p'} a^r b^s (\mathbf{g}_r \times \mathbf{g}_s) \cdot \mathbf{g}_t = h^{t}_{p'} c_t.$$

For axial vector we obtain a standard law of transformation for covariant coordinates of vectors.

Thus in the oriented SR there are two kinds of objects: polar objects and axial objects.

Definition. *Objects, which are independent of the choice of orientation of SR, are called the polar objects; objects, which depend on the choice of orientation of SR and are multiplied by (-1) under changing of orientation of SR, are called the axial objects.*

In according with the definition axial objects may be represented by scalars, vectors and tensors of any rank. The well known examples of axial objects are: the mixed product of three polar vectors $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, vector of angular velocity, the Levi - Civita tensor $\mathbf{E} \times \mathbf{E}$ and so on.

3 Modified Definition of Orthogonal Transformation

In oriented space the definition of orthogonal transformation (3) must be slightly changed.

Definition. *The orthogonal transformation of the scalar g, of the vector $\mathbf{a}$, of the second-rank tensor $\mathbf{A}$ and of the n-rank tensor $\mathbf{D}$ are defined by formulae*

$$g' \equiv (\det\mathbf{Q})^{\alpha} g, \quad \mathbf{a}' \equiv (\det\mathbf{Q})^{\alpha}\mathbf{Q}\cdot\mathbf{a}, \quad \mathbf{A}' \equiv (\det\mathbf{Q})^{\alpha}\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^{T},$$
$$\mathbf{D}' \equiv (\det\mathbf{Q})^{\alpha}\mathbf{Q}^{n}\odot\mathbf{D}, \quad (10)$$

where $\alpha = 0$ for polar objects, $\alpha = 1$ for axial objects.

The definition (10) coincides with the classical definition (3) for the polar tensors. Correctness of the definition (10) can be shown by means of simple examples. Let us consider the axial scalar $\psi = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, where all vectors are polar ones. Then the orthogonal transformation of the scalar ψ may me found directly

$$\psi' = \mathbf{a}' \cdot (\mathbf{b}' \times \mathbf{c}') = (\mathbf{Q}\cdot\mathbf{a}) \cdot [(\mathbf{Q}\cdot\mathbf{b}) \times (\mathbf{Q}\cdot\mathbf{c})] =$$
$$= (\mathbf{a}\cdot\mathbf{Q}^{T}) \cdot [(\det\mathbf{Q})\mathbf{Q}\cdot(\mathbf{b}\times\mathbf{c})] = (\det\mathbf{Q})\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\det\mathbf{Q})\psi,$$

where the identity

$$(\mathbf{Q}\cdot\mathbf{b}) \times (\mathbf{Q}\cdot\mathbf{c}) = (\det\mathbf{Q})\,\mathbf{Q}\cdot(\mathbf{b}\times\mathbf{c})$$

was used.

As a result we obtain the definition (10). The vector product of two polar vectors is a typical example of an axial vector. In this case the direct definition is possible as well

$$\mathbf{c}' = \mathbf{a}' \times \mathbf{b}' = (\mathbf{Q}\cdot\mathbf{a}) \times (\mathbf{Q}\cdot\mathbf{b}) = (\det\mathbf{Q})\,\mathbf{Q}\cdot(\mathbf{a}\times\mathbf{b}) = (\det\mathbf{Q})\,\mathbf{Q}\cdot\mathbf{c}.$$

By this way the definition (10) may be derived for a tensor of any rank.

4 Symmetry Groups of Tensors

Definition. *The sets of the orthogonal solutions of the equations*

$$(\det\mathbf{Q})^{\alpha}\psi = \psi,\ (\det\mathbf{Q})^{\alpha}\mathbf{Q}\cdot\mathbf{a} = \mathbf{a},\ (\det\mathbf{Q})^{\alpha}\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^{T} = \mathbf{A},\ (\det\mathbf{Q})^{\alpha}\mathbf{Q}^{n}\odot\mathbf{D} = \mathbf{D} \quad (11)$$

are called the symmetry groups (SG) of the scalar ψ*, the vector* $\mathbf{a}$*, the second rank tensor* $\mathbf{A}$ *and the n-rank tensor* $\mathbf{D}$ *correspondingly, where* ψ*,* $\mathbf{a}$*,* $\mathbf{A}$ *and* $\mathbf{D}$ *are given, orthogonal tensors* $\mathbf{Q}$ *must be found.*

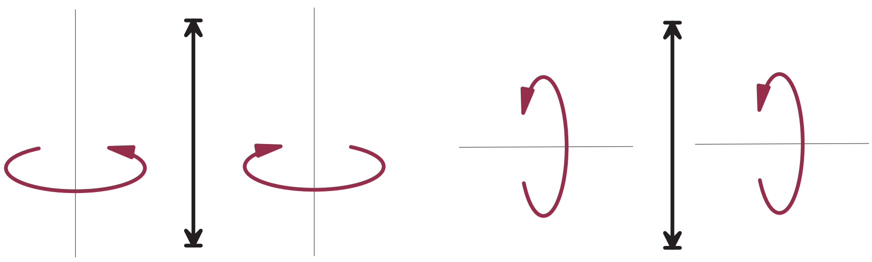
Let us consider some examples.

Symmetry group of vector. For a polar vector $\mathbf{a}$ the SG contain tensors (2), where $\mathbf{m} = \mathbf{a}/|\mathbf{a}|$, and tensors (1), where $\mathbf{n}$ is any unit vector such that $\mathbf{n} \cdot \mathbf{a} = 0$.

In order to establish SG of an axial vector $\mathbf{a}$ the orthogonal solutions of the equation

$$(\det \mathbf{Q})\mathbf{Q} \cdot \mathbf{a} = \mathbf{a}$$

must be found. It is easy to see that the SG of the axial vector $\mathbf{a}$ contains tensors (2), where $\mathbf{m} \times \mathbf{a} = 0$, and tensors (1), where $\mathbf{n} \times \mathbf{a} = 0$. The correctness of this result may be easy seen at the Fig. 2.



a). The mirror reflection from the plane parallel to the axis of spin-vector

b). The mirror reflection from the plane orthogonal to the axis of spin-vector

Figure 2: On a symmetry of an axial vector

Thus the symmetry elements of polar and axial vectors are different under the mirror reflections.

Usually it is not difficult to find the SG of any given tensor. However in the theory of constitutive equations the solution of the inverse problem is much more important. For this it is necessary to find the structure of a tensor with given elements of symmetry.

Second rank tensor with one plane of the mirror symmetry. Polar and axial tensors with the same elements of symmetry have different structures. Let, for example, the mirror reflection $\mathbf{E} - 2\mathbf{m} \otimes \mathbf{m}$ belongs to the SG of a polar tensor $\mathbf{A}$ and an axial tensor $\mathbf{B}$. It is possible if and only if these tensors have the form

$$\mathbf{A} = A_{11}\mathbf{m} \otimes \mathbf{m} + A_{22}\mathbf{n} \otimes \mathbf{n} + A_{23}\mathbf{n} \otimes \mathbf{p} + A_{32}\mathbf{p} \otimes \mathbf{n} + A_{33}\mathbf{p} \otimes \mathbf{p},$$

$$\mathbf{B} = B_{12}\mathbf{m} \otimes \mathbf{n} + B_{13}\mathbf{m} \otimes \mathbf{p} + B_{21}\mathbf{n} \otimes \mathbf{m} + B_{31}\mathbf{p} \otimes \mathbf{m},$$

where A_{ik} are absolute scalars and B_{ik} are axial scalars, $\mathbf{m}$, $\mathbf{n}$, $\mathbf{p}$ is an orthogonal basis. If there are two planes of mirror symmetry with unit normals $\mathbf{m}$ and $\mathbf{n}$, then

$$\mathbf{A} = A_{11}\mathbf{m} \otimes \mathbf{m} + A_{22}\mathbf{n} \otimes \mathbf{n} + A_{33}\mathbf{p} \otimes \mathbf{p}, \quad \mathbf{B} = B_{12}\mathbf{m} \otimes \mathbf{n} + B_{21}\mathbf{n} \otimes \mathbf{m}. \tag{12}$$

Example: naturally twisted rods. *The specific energy of thin elatic rods is determined by quadratic form*

$$\mathcal{U} = \frac{1}{2}\mathbf{e}\cdot\mathbf{A}\cdot\mathbf{e} + \mathbf{e}\cdot\mathbf{B}\cdot\boldsymbol{\kappa} + \frac{1}{2}\boldsymbol{\kappa}\cdot\mathbf{C}\cdot\boldsymbol{\kappa}, \qquad (13)$$

where the vectors of deformation are defined by the expressions

$$\mathbf{e} = \mathbf{u}' + \mathbf{p}\times\boldsymbol{\varphi}, \quad \boldsymbol{\kappa} = \boldsymbol{\varphi}'.$$

Let vectors $\mathbf{m}$ and $\mathbf{n}$ be principle directions of the rod cross section. Let tensors $\mathbf{E} - 2\mathbf{m}\otimes\mathbf{m}$ and $\mathbf{E} - 2\mathbf{n}\otimes\mathbf{n}$ be elements of symmetry for the cross section. If we apply the classical theory of symmetry, then we obtain that in such a case the tensors of elasticity $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in (13) have the form of the tensor $\mathbf{A}$ in (12). This is nonsense from physical point of view. If we use the modified theory of symmetry, then only the polar tensors of elasticity $\mathbf{A}$ and $\mathbf{C}$ in (13) have the form of the tensor $\mathbf{A}$ in (12) but the axial tensor $\mathbf{B}$ in (13) has the form of the tensor $\mathbf{B}$ in (12). Let $\mathbf{p}$ be the unit normal vector to the cross section. As a rule tensor $\mathbf{E} - 2\mathbf{p}\otimes\mathbf{p}$ belongs to the SG of the rod. In such a case the axial tensor of elasticity $\mathbf{B}$ in (13) is equal to zero. However for naturally twisted rods (for example, drills) the tensor $\mathbf{E} - 2\mathbf{p}\otimes\mathbf{p}$ does not belong to the SG of the rod. Because of this the axial tensor of elasticity $\mathbf{B}$ in (13) has the form of the tensor $\mathbf{B}$ in (12). This result can not be obtained with the help of the classical theory of symmetry.

Definition. *The n-rank tensor is called isotropic one if its group of symmetry contains all orthogonal tensors.*

There is one polar isotropic 2nd-rank tensor $f\mathbf{E}$, where f is an absolute scalar. There are no axial isotropic tensors of the 2nd-rank. There are no polar isotropic tensors of the 3d-rank. However there is one axial isotropic 3d-rank tensor $f\mathbf{E}\times\mathbf{E}$, where f is an absolute scalar. Indeed, according to the definition (10) we have

$$(\mathbf{E}\times\mathbf{E})' \equiv (\mathbf{g}^m\otimes\mathbf{g}_m\times\mathbf{g}^n\otimes\mathbf{g}_n)' \equiv (\det\mathbf{Q})\mathbf{Q}\cdot\mathbf{g}^m\otimes\mathbf{Q}\cdot(\mathbf{g}_m\times\mathbf{g}^n)\otimes\mathbf{Q}\cdot\mathbf{g}_n =$$

$$= [\mathbf{Q}\cdot(\mathbf{g}^m\otimes\mathbf{g}_m)\cdot\mathbf{Q}^T]\times[\mathbf{Q}\cdot(\mathbf{g}^n\otimes\mathbf{g}_n)\cdot\mathbf{Q}^T] = [\mathbf{Q}\cdot\mathbf{E}\cdot\mathbf{Q}^T]\times[\mathbf{Q}\cdot\mathbf{E}\cdot\mathbf{Q}^T] = \mathbf{E}\times\mathbf{E}.$$

The tensor $f\mathbf{E}\times\mathbf{E}$ is supposed to be non-isotropic under the conventional approach. This fact is important in the theory of piezoelectricity.

5 Orthogonal invariants and theorem on basis

Let there be given the finite collection of the tensors (6).

Definition. *A scalar-valued tensor function*

$$F = F(\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_m, \mathbf{A}_1, \mathbf{A}_2, \dots \mathbf{A}_n)$$

is called an orthogonal invariant of the collection (6) if the equality

$$F(\mathbf{a}_1', \dots, \mathbf{a}_m', \mathbf{A}_1', \dots, \mathbf{A}_n') = (\det\mathbf{Q})^\alpha F(\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{A}_1, \dots \mathbf{A}_n), \qquad (14)$$

holds for all orthogonal tensors $\mathbf{Q}$; the quantities with primes are defined by (10); $\alpha = 0$, if values of F are absolute scalars, and $\alpha = 1$, if values of F are axial scalars.

Let us consider the function

$$\psi(\mathbf{a}, \mathbf{b}, \mathbf{c}) \equiv (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}, \qquad (15)$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are polar vectors. Accordingly to classical definition (5) the function ψ is not orthogonal invariant. In accordance with the definition (14) the function ψ is orthogonal invariant since

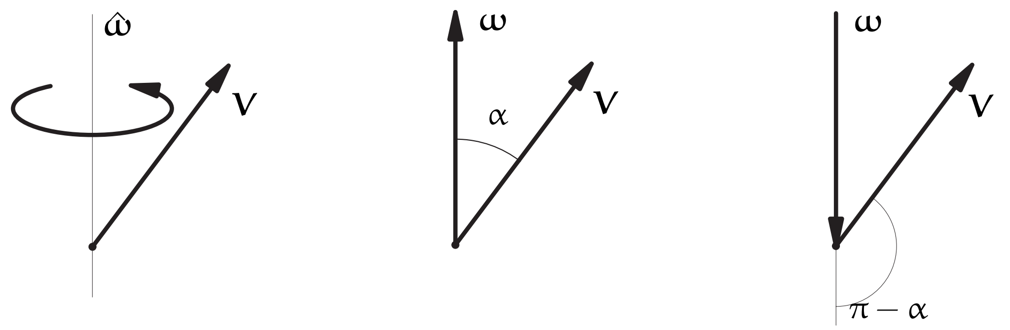
$$\psi(\mathbf{a}', \mathbf{b}', \mathbf{c}') = (\mathbf{a}' \times \mathbf{b}') \cdot \mathbf{c}' = (\det \mathbf{Q})(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\det \mathbf{Q})\psi(\mathbf{a}, \mathbf{b}, \mathbf{c}).$$

Another example is the scalar product of polar $\mathbf{V}$ and axial $\boldsymbol{\omega}$ vectors

$$\psi(\mathbf{V}, \boldsymbol{\omega}) \equiv \mathbf{V} \cdot \boldsymbol{\omega}. \qquad (16)$$

With respect to definition (5) this function is not an orthogonal invariant. In accordance with the definition (14) the function (16) is orthogonal invariant. From the pure mathematical point of view this is a question of the definitions and there is no subject for discussions. However the situation is quiet different if we consider the problem from physical point of view.

Let us consider an one-spin particle — see Figure 3. An one-spin particle, shown with using



a). One-spin particle: a physical object

b). One-spin particle: the mathematical image in the right-oriented SR

c). One-spin particle: the mathematical image in the left-oriented SR

Figure 3: One spin-particle: physical and mathematical images

of spin-vector, is presented at the Figure 3a. This image does not depend on the SR orientation. The mathematical image, obtained with using of an axial vector, is shown on the Figure 3b in the right-oriented SR and on the Figure 3c in the left-oriented SR. It is seen that

$$(\mathbf{V} \cdot \boldsymbol{\omega})_R \equiv V\omega \cos\alpha = -(\mathbf{V} \cdot \boldsymbol{\omega})_L \equiv -V\omega\cos(\pi - \alpha).$$

Let us note that the nature and physical objects, for example spin-vectors, know nothing about orientation of SR. Axial vectors are some mathematical inventions and they feel the change of the SR orientation. The scalar product $(\mathbf{V} \cdot \boldsymbol{\omega})_R$ in the right-oriented SR is not equal to the scalar product $(\mathbf{V} \cdot \boldsymbol{\omega})_L$ in the left-oriented SR. However both of them correspond to the single physical object. By this reason the scalar product (16) must be called an invariant. This fact is taken into account by the definition (14).

Now we are able to discuss the problem of invariants. It is obvious that Hilbert's theorem is valid both for the definition (14) of orthogonal invariants of the collection (6) and for the classical definition (5). However Hilbert's theorem says nothing about the number of functionally independent invariants consisting the basis. From the pure physical point of view the basis dimension may be found by simple calculation. The result of this calculation leads to the following statement.

Theorem. *The dimension* N_* *of the invariant basis of the collection (6) is related with the number* N *of independent coordinates of objects in (6) by the next formulae*

$$N_* = 1 \ \textit{when} \ m = 1,\, n = 0; \quad N_* = N - 3 \tag{17}$$

in all other cases.

The proof of the theorem will be given in what follows. Let us note that well-known Rivlin's theorem states that $N_* = N - 2$.

6 Generic Equation for Invariants

The definition (14) of an invariant of the collection (6) contain an arbitrary orthogonal tensor $\mathbf{Q}$. In what follows without loss of generality we may suppose that tensor $\mathbf{Q}$ is a proper orthogonal tensor since an arbitrary orthogonal tensor $\mathbf{Q}$ can be represented as composition $\mathbf{Q} = (-\mathbf{E}) \cdot \mathbf{P}$, where $\mathbf{P}$ is a proper orthogonal tensor. The inversion transformation may be taken into account later. In such a case let us consider a continuous set of orthogonal tensors $\mathbf{Q}(\tau)$, depending on real parameter τ. It is easy to prove that there exist an axial vector $\boldsymbol{\omega}(\tau)$ satisfying the equation

$$\frac{d}{d\tau}\mathbf{Q}(\tau) = \boldsymbol{\omega}(\tau) \times \mathbf{Q}(\tau), \quad \mathbf{Q}(0) = \mathbf{E}, \qquad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0 \neq \mathbf{0}. \tag{18}$$

If the orthogonal tensor $\mathbf{Q}(\tau)$ in (14) depend on the parameter τ, then we may differentiate both sides of (14) with respect to τ. In such a case we obtain the following equation

$$\sum_{i=1}^{m} \frac{\partial F}{\partial \mathbf{a}_i'} \cdot \frac{d\mathbf{a}_i'}{d\tau} + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial \mathbf{A}_i'}\right)^T \cdot\cdot \frac{d\mathbf{A}_i'}{d\tau} = 0. \tag{19}$$

The derivatives of a scalar function with respect to vector and tensor arguments is defined by the rule

$$dF = \sum_{i=1}^{m} \frac{\partial F}{\partial \mathbf{a}_i} \cdot d\mathbf{a}_i + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial \mathbf{A}_i}\right)^T \cdot\cdot d\mathbf{A}_i. \tag{20}$$

An example*. Let there be given a scalar function*

$$F(\mathbf{a}, \mathbf{b}, \mathbf{A}) = \mathbf{a} \cdot \mathbf{A} \cdot \mathbf{b} \quad \Rightarrow \quad dF = (\mathbf{A} \cdot \mathbf{b}) \cdot d\mathbf{a} + (\mathbf{a} \cdot \mathbf{A}) \cdot d\mathbf{b} + (\mathbf{a} \otimes \mathbf{b})^T \cdot\cdot d\mathbf{A}.$$

According to (20) one has

$$\frac{\partial F}{\partial \mathbf{a}} = \mathbf{A} \cdot \mathbf{b}, \quad \frac{\partial F}{\partial \mathbf{b}} = \mathbf{a} \cdot \mathbf{A}, \quad \frac{\partial F}{\partial \mathbf{A}} = \mathbf{a} \otimes \mathbf{b}.$$

Making use of equality (18), the derivatives may be calculated

$$\frac{d\mathbf{a}'_i}{d\tau} = \boldsymbol{\omega}(\tau) \times \mathbf{a}'_i, \quad \frac{d\mathbf{A}'_i}{d\tau} = \boldsymbol{\omega}(\tau) \times \mathbf{A}'_i - \mathbf{A}'_i \times \boldsymbol{\omega}(\tau).$$

Let us take into account the equalities

$$\mathbf{a}'_i(0) = \mathbf{a}_i, \quad \mathbf{A}'_i(0) = \mathbf{A}_i.$$

Let τ in the equality (19) be equal to zero. Using the equalities written above we obtain

$$\sum_{i=1}^{n} \left(\frac{\partial F}{\partial \mathbf{A}_i}\right)^T \cdot\cdot (\boldsymbol{\omega}_0 \times \mathbf{A}_i - \mathbf{A}_i \times \boldsymbol{\omega}_0) + \sum_{i=1}^{m} \frac{\partial F}{\partial \mathbf{a}_i} \cdot (\boldsymbol{\omega}_0 \times \mathbf{a}_i) = 0. \tag{21}$$

The equation (21) is the linear homogenous equation in the partial derivatives of the first order. This equation must be valid for any vector $\boldsymbol{\omega}_0$. Because of this the equation (21) is equivalent to the three scalar equations. Any scalar invariant of the collection (6) must be solution of the equation (21). Instead of the eq. (21) it is more convenient to use an equation obtained from the eq. (21) by dividing on the modulus of vector $\boldsymbol{\omega}_0$

$$\sum_{i=1}^{n} \left(\frac{\partial F}{\partial \mathbf{A}_i}\right)^T \cdot\cdot (\mathbf{m} \times \mathbf{A}_i - \mathbf{A}_i \times \mathbf{m}) + \sum_{i=1}^{m} \frac{\partial F}{\partial \mathbf{a}_i} \cdot (\mathbf{m} \times \mathbf{a}_i) = 0, \tag{22}$$

which must be valid for any unit vector $\mathbf{m}$.

In what follows the equation (22) will be called the generic equation for the invariants. The unit vector $\mathbf{m}$ may be excluded from the equation (22). In such a case instead of the scalar equation (22) we obtain the vector equation

$$\sum_{i=1}^{n} \left[\left(\frac{\partial F}{\partial \mathbf{A}_i}\right)^T \cdot \mathbf{A}_i + \mathbf{A}_i \cdot \left(\frac{\partial F}{\partial \mathbf{A}_i}\right)^T\right]_\times + \sum_{i=1}^{m} \frac{\partial F}{\partial \mathbf{a}_i} \times \mathbf{a}_i = \mathbf{0}, \quad (\mathbf{a} \otimes \mathbf{b})_\times \equiv \mathbf{a} \times \mathbf{b}, \tag{23}$$

where the vector $\mathbf{A}_\times$ is called a vector invariant of the 2nd-rank tensor $\mathbf{A}$. The vector equation (23) is equivalent to the three scalar equations. Any orthogonal invariant must satisfy this three equations. However in general case not all of these equations are independent. If the collection (6) contains more than one vector, then all three equations are independent.

The equation (23) is a system of three linear equation in partial derivatives. Any scalar invariant of the collection (6) must be solution of the equation (21). And conversely, any solution of equation (23) is the invariant of the collection (6). Coordinates of vectors and tensors of the set (6) are independent variables. Therefore this equation is defined in the space of dimension $N = 3m + 6n$. The function $F(\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_m, \quad \mathbf{A}_1, \mathbf{A}_2, \dots \mathbf{A}_n)$ depends on N arguments. The theory of the equations in the partial derivatives of the first order is well developed. It may be said that each of the scalar equation of the system (23) decreases the number of independent variables on one. The number of the rest independent variables is the number of functionally independent invariants. Thus the dimension of the invariant basis is equal to $N - q$, where $2 \le q \le 3$ is the number of independent equation of the system (23). If $N = 3$, then $q = 2$. If $N > 3$, then $q = 3$.

Below the application of this theorem will be considered.

7 The basis invariant of vector

In this case the equation (23) takes a simple form

$$\frac{\partial F}{\partial \mathbf{a}} \times \mathbf{a} = 0 \quad \Leftrightarrow$$

$$a_1 \frac{\partial F}{\partial a_2} - a_2 \frac{\partial F}{\partial a_1} = 0,\ a_2 \frac{\partial F}{\partial a_3} - a_3 \frac{\partial F}{\partial a_2} = 0,\ a_1 \frac{\partial F}{\partial a_3} - a_3 \frac{\partial F}{\partial a_1} = 0,$$

where a_i are coordinates of vector $\mathbf{a}$ with respect to some orthogonal basis.

The last equation is a consequence of two previous equations. Because of this there are only two independent equations

$$a_1 \frac{\partial F}{\partial a_2} - a_2 \frac{\partial F}{\partial a_1} = 0, \quad a_2 \frac{\partial F}{\partial a_3} - a_3 \frac{\partial F}{\partial a_2} = 0. \tag{24}$$

The function F must satisfy two independent equation (24) in partial derivatives of the first order. We have to find a general solution of the first equation. After that this general solution must be obeyed to the second equation.

In order to find a general solution of the first equation we have to write down the characteristic system for this equation

$$\frac{da_1}{ds} = -a_2,\ \frac{da_2}{ds} = a_1 \ \Rightarrow\ \frac{d}{ds}\left(a_1^2 + a_2^2\right) = 0, \tag{25}$$

where $a_i(s)$ is the parametric form of a curve in 3D-space of coordinates a_i, s is the parameter. The linear system (25) of the second order has not more then one independent integral. Any function of this integral is an integral of (25) as well. The integral shown by the last equality in (25) may be taken as such independent integral. Thus a general solution of the first equation from (25) is given by expression

$$F(\mathbf{a}) = F(a_1,\ a_2,\ a_3) = f(a_1^2 + a_2^2,\ a_3).$$

Substituting this solution into the second equation from system (24), we obtain

$$\frac{\partial f}{\partial a_3} - 2a_3 \frac{\partial f}{\partial q} = 0, \quad q \equiv a_1^2 + a_2^2.$$

The characteristic system for this equation has a form

$$\frac{dq}{ds} = -2\,a_3, \quad \frac{da_3}{ds} = 1 \quad \Rightarrow \quad \frac{d}{ds}\left(q + a_3^2\right) = 0.$$

This system of the second order has the only independent integral. One may choose the integral $q + a_3^2$. Therefore any orthogonal invariant may be expressed as a function of the modulus of vector $\mathbf{a}$

$$F(\mathbf{a}) = f(q,\ a_3) = g(\mathbf{a} \cdot \mathbf{a}).$$

In this case the basic theorem has been almost proved. Of course this result is well-known and was obtained by O. Cauchy.

Let us obtain this result by direct coordinate-free approach. The equation (22) and corresponding to it the characteristic system take a form

$$\frac{\partial F}{\partial \mathbf{a}} \cdot (\mathbf{m} \times \mathbf{a}) = 0, \quad \frac{d\mathbf{a}}{ds} = \mathbf{m} \times \mathbf{a}. \qquad (26)$$

The vector characteristic equation (26) has the third order and two independent integrals

$$\mathbf{a} \cdot \mathbf{a} = \mathrm{const}, \quad \mathbf{m} \cdot \mathbf{a} = \mathrm{const}.$$

The second integral depends on arbitrary vector $\mathbf{m}$ and must be taken off. The coordinate-free approach is much shorter and will be used below.

In order to finish the proof of theorem we have to show that two vectors with the same moduli may be transformed from one to other by means of pure turn. Let there be given two vectors $\mathbf{a}$ and $\mathbf{b}$, whose moduli and kinds are the same. A general transformation of a vector, conserving its modulus, is given by

$$\mathbf{a} = (\det \mathbf{Q})^{\alpha} \mathbf{Q} \cdot \mathbf{b}, \quad \mathbf{Q} \cdot \mathbf{Q}^{T} = \mathbf{E}, \quad \det \mathbf{Q} = \pm 1,$$

where $\alpha = 0$ for polar vectors and $\alpha = 1$ for axial vectors. It is not what we wont. In ***I-problem*** the tensor $\mathbf{Q}$ must be proper orthogonal tensor. Let us note that the tensor $\mathbf{Q}$ in the last equality is not completely defined. Indeed this equality may be rewritten in the equivalent form

$$\mathbf{a} = (\det \mathbf{Q})^{\alpha} \mathbf{Q} \cdot (\det \mathbf{S})^{\alpha} \mathbf{S} \cdot \mathbf{b}, \quad (\det \mathbf{S})^{\alpha} \mathbf{S} \cdot \mathbf{b} = \mathbf{b},$$

where tensor $\mathbf{S}$ belongs to SG of vector $\mathbf{b}$. Thus we have

$$\mathbf{a} = \mathbf{Q}_{*} \cdot \mathbf{b}, \quad \mathbf{Q}_{*} \equiv \det(\mathbf{Q} \cdot \mathbf{S})^{\alpha}\, \mathbf{Q} \cdot \mathbf{S}, \quad \det \mathbf{Q}_{*} = [\det(\mathbf{Q} \cdot \mathbf{S})]^{1+\alpha} = 1.$$

If $\alpha = 1$, then the last condition is valid identically. If $\alpha = 0$, then it may be satisfied by appropriate choice of the tensor $\mathbf{S}$. For example, if $\det \mathbf{Q} = -1$, then we may take $\mathbf{S} = \mathbf{E} - 2\mathbf{m} \otimes \mathbf{m}$, where $\mathbf{m} \cdot \mathbf{b} = 0, \quad |\mathbf{m}| = 1$.

8 Basic invariants for a set of three vectors

Let us find the minimally complete set of basic invariants for a collection of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The equation (22) for invariant $F(\mathbf{a}, \mathbf{b}, \mathbf{c})$ takes the form

$$\frac{\partial F}{\partial \mathbf{a}} \cdot (\mathbf{m} \times \mathbf{a}) + \frac{\partial F}{\partial \mathbf{b}} \cdot (\mathbf{m} \times \mathbf{b}) + \frac{\partial F}{\partial \mathbf{c}} \cdot (\mathbf{m} \times \mathbf{c}) = 0.$$

The characteristic system for this equation is

$$\frac{d\mathbf{a}}{ds} = \mathbf{m} \times \mathbf{a}, \quad \frac{d\mathbf{b}}{ds} = \mathbf{m} \times \mathbf{b}, \quad \frac{d\mathbf{c}}{ds} = \mathbf{m} \times \mathbf{c}. \qquad (27)$$

General solution of (27) is given by

$$\mathbf{a}(s) = \mathbf{Q}(s\mathbf{m}) \cdot \mathbf{a}_0, \ \mathbf{b}(s) = \mathbf{Q}(s\mathbf{m}) \cdot \mathbf{b}_0, \ \mathbf{c}(s) = \mathbf{Q}(s\mathbf{m}) \cdot \mathbf{c}_0, \qquad (28)$$

where vectors $\mathbf{a}_0$, $\mathbf{b}_0$, $\mathbf{c}_0$ are arbitrary constant vectors. The system (27) of ninth order has not more than eight integrals. In order to find these integrals, it is necessary to exclude the variable s and the vector $\mathbf{m}$ from (28). It is easy to build up ten integrals. Three of them depend on arbitrary vector $\mathbf{m}$

$$\mathbf{m} \cdot \mathbf{a} = \mathbf{m} \cdot \mathbf{a}_0, \quad \mathbf{m} \cdot \mathbf{b} = \mathbf{m} \cdot \mathbf{b}_0, \quad \mathbf{m} \cdot \mathbf{c} = \mathbf{m} \cdot \mathbf{c}_0.$$

These integrals must be ignored.

Seven invariants are determined by the next integrals

$$I_1 = \mathbf{a} \cdot \mathbf{a}, \quad I_2 = \mathbf{b} \cdot \mathbf{b}, \quad I_3 = \mathbf{c} \cdot \mathbf{c}, \quad I_4 = \mathbf{a} \cdot \mathbf{b}, \quad I_5 = \mathbf{a} \cdot \mathbf{c}, \quad I_6 = \mathbf{b} \cdot \mathbf{c}, \tag{29}$$

$$I_7 = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}). \tag{30}$$

Invariants $I_1 - I_7$ are not independent since there is an obvious relation

$$I_7^2 \equiv \begin{vmatrix} I_1 & I_4 & I_5 \\ I_4 & I_2 & I_6 \\ I_5 & I_6 & I_3 \end{vmatrix}.$$

Therefore seven integrals (29) and (30) may be expressed in terms of six functionally independent integrals.

This example clearly shows the distinction between classical approach and the approach under consideration. Accordingly to classical definitions the axial scalar is not invariant. Because of this some scalars must be excluded from the list (29). For example, let vectors $\mathbf{a}$ and $\mathbf{b}$ be polar ones but $\mathbf{c}$ is an axial vector. In such a case the invariants I_5 and I_6 are the axial scalars and must be excluded from the list (29). That means that the classical problem of invariants has no solution. If all three vectors are polar ones, then from the classical point of view the invariant I_7 must be excluded from the list (29)–(30). However the fixation of the invariants $I_1 - I_6$ does not fix the triple of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as a rigid whole. In such a case the ***I-problem*** has no solution. Indeed let us consider the two triples of vectors: $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and

$$\mathbf{a}, \quad \mathbf{b}, \quad (\mathbf{E} - 2\mathbf{k} \otimes \mathbf{k}) \cdot \mathbf{c}, \quad \mathbf{k} = \mathbf{a} \times \mathbf{b} / |\mathbf{a} \times \mathbf{b}|.$$

This triple of vectors is formed from polar vectors. Invariants $I_1 - I_6$ are the same for both triples of vectors. Nevertheless the second triple can not be obtained from the first triple by rigid rotation. Invariants (29)–(30) for these triples are different.

In order to solve the ***I-problem*** we must prove the existing of the proper orthogonal tensor $\mathbf{P}$ such that the relations (8) holds good. Let there be given two triples of vectors

$$\mathbf{a}, \mathbf{b}, \mathbf{c} \qquad \text{and} \qquad \mathbf{a}_* = a\mathbf{m}, \quad \mathbf{b}_* = b\mathbf{n}, \quad \mathbf{c}_* = c\mathbf{p}.$$

Invariants $I_1 - I_3$ for these triples are supposed to be the same. In such a case we have

$$\mathbf{a} = a\mathbf{Q}_a \cdot \mathbf{m}, \quad \mathbf{b} = b\mathbf{Q}_b \cdot \mathbf{n}, \quad \mathbf{c} = c\mathbf{Q}_c \cdot \mathbf{p}, \tag{31}$$

where $\mathbf{Q}_a$, $\mathbf{Q}_b$, $\mathbf{Q}_c$ are orthogonal tensors. The coincidence of invariants $I_4 - I_6$ gives equations for determining of tensors $\mathbf{Q}_a$, $\mathbf{Q}_b$, $\mathbf{Q}_c$

$$\mathbf{m} \cdot \mathbf{Q}_a^T \cdot \mathbf{Q}_b \cdot \mathbf{n} = \mathbf{m} \cdot \mathbf{n}, \quad \mathbf{m} \cdot \mathbf{Q}_a^T \cdot \mathbf{Q}_c \cdot \mathbf{p} = \mathbf{m} \cdot \mathbf{p}, \quad \mathbf{n} \cdot \mathbf{Q}_b^T \cdot \mathbf{Q}_c \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{p}. \tag{32}$$

From (32) the next equalities may be obtained

$$\mathbf{Q}_a^T \cdot \mathbf{Q}_b = \mathbf{S}_{(m,1)} \cdot \mathbf{S}_{(n,1)}, \quad \mathbf{Q}_a^T \cdot \mathbf{Q}_c = \mathbf{S}_{(m,2)} \cdot \mathbf{S}_{(p,1)}, \quad \mathbf{Q}_b^T \cdot \mathbf{Q}_c = \mathbf{S}_{(n,2)} \cdot \mathbf{S}_{(p,2)}, \tag{33}$$

where orthogonal tensors $\mathbf{S}_{(m,k)}, \mathbf{S}_{(n,k)}, \mathbf{S}_{(p,k)}$ are some elements of symmetry of the vectors $\mathbf{m}$, $\mathbf{n}$ and $\mathbf{p}$ correspondingly. The equalities (33) lead to the restrictions

$$\mathbf{Q}_b^T \cdot \mathbf{Q}_c = \mathbf{Q}_b^T \cdot \mathbf{Q}_a \cdot \mathbf{Q}_a^T \cdot \mathbf{Q}_c = \mathbf{S}_{(n,1)}^T \cdot \mathbf{S}_{(m,1)}^T \cdot \mathbf{S}_{(m,2)} \cdot \mathbf{S}_{(p,1)} = \mathbf{S}_{(n,2)} \cdot \mathbf{S}_{(p,2)}. \tag{34}$$

Now the equalities (31) takes a form

$$\mathbf{a} = a\mathbf{Q}_a \cdot \mathbf{m}, \ \mathbf{b} = b\mathbf{Q}_a \cdot \mathbf{S}_{(m,1)} \cdot \mathbf{n}, \ \mathbf{c} = c\mathbf{Q}_a \cdot \mathbf{S}_{(m,2)} \cdot \mathbf{p},$$

where tensors $\mathbf{S}_{(m,1)}, \mathbf{S}_{(m,2)}$ are two arbitrary elements of symmetry of $\mathbf{m}$. Invariants $I_1 - I_6$ of two triples under consideration for any orthogonal tensor $\mathbf{Q}_a$ coincide. However the distinction between these triples of vectors can not be reduced to the rigid rotation. If the invariants I_7 for these triple of vectors coincide, then the equation

$$\det\mathbf{Q}_a \det\mathbf{S}_{m,1} (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{S}_{(m,1)}^T \cdot \mathbf{S}_{(m,2)} \cdot \mathbf{p} = (\mathbf{m} \times \mathbf{n}) \cdot \mathbf{p}.$$

must be valid. It will be so if and only if

$$\det\mathbf{Q}_a \det\mathbf{S}_{(m,1)} = 1, \quad \mathbf{S}_{(m,1)} = \mathbf{S}_{(m,2)}.$$

Finally we obtain

$$\mathbf{a} = \mathbf{P} \cdot \mathbf{a}_*, \quad \mathbf{b} = \mathbf{P} \cdot \mathbf{b}_*, \quad \mathbf{c} = \mathbf{P} \cdot \mathbf{c}_*, \quad \mathbf{P} = \mathbf{Q}_a \cdot \mathbf{S}_{(m,1)},$$

where $\mathbf{Q}_a$ is an arbitrary orthogonal tensor, the tensor $\mathbf{S}_{(m,1)}$ is a such element of symmetry of vector $\mathbf{a}$, that tensor $\mathbf{P}$ is a proper orthogonal tensor. Thus we obtain the solution of the ***I-problem***.

9 Basic invariants of a symmetric second-rank tensor

It is known that any orthogonal invariant of the symmetric second-rank tensor may be represented as a function of its principle invariants. Let us obtain this result by means of our approach. Let $F(\mathbf{A})$ be some orthogonal invariant of $\mathbf{A}$. Then it must satisfy the equation (22) for $m = 0$, $n = 1$, which takes a form

$$\left(\frac{\partial F}{\partial \mathbf{A}}\right)^T \cdot\cdot (\mathbf{m} \times \mathbf{A} - \mathbf{A} \times \mathbf{m}) = 0.$$

Let us write down the characteristic system for (9)

$$\frac{d\mathbf{A}(s)}{ds} = \mathbf{m} \times \mathbf{A}(s) - \mathbf{A}(s) \times \mathbf{m}. \tag{35}$$

We obtain the sixth-order system which has exactly five independent integrals. However two of them depends on arbitrary unit vector $\mathbf{m}$ and may be ignored. A general solution of (35) is given by expression

$$\mathbf{A}(s) = \mathbf{Q}(s\mathbf{m}) \cdot \mathbf{A}_0 \cdot \mathbf{Q}^T(s\mathbf{m}), \tag{36}$$

where $\mathbf{A}_0$ is the tensor $\mathbf{A}$ in some fixed position, $\mathbf{Q}$ is a turn-tensor.

Excluding tensor $\mathbf{Q}$ from the solution (36) we obtain five independent integrals of (35)

$$I_1 = \operatorname{tr}\mathbf{A} = \operatorname{tr}\mathbf{A}_0, \quad I_2 = \operatorname{tr}\mathbf{A}^2 = \operatorname{tr}\mathbf{A}_0^2, \quad I_3 = \operatorname{tr}\mathbf{A}^3 = \operatorname{tr}\mathbf{A}_0^3,$$

$$I_4 = \mathbf{m}\cdot\mathbf{A}\cdot\mathbf{m} = \mathbf{m}\cdot\mathbf{A}_0\cdot\mathbf{m}, \quad I_5 = \mathbf{m}\cdot\mathbf{A}^2\cdot\mathbf{m} = \mathbf{m}\cdot\mathbf{A}_0^2\cdot\mathbf{m}.$$

Any integral of (35) may be represented as a function of these integrals: $f(I_1, I_2, I_3, I_4, I_5)$. It is easy to see that the function $f(I_1, I_2, I_3, I_4, I_5)$ is an orthogonal invariant $F(\mathbf{A})$ of the tensor $\mathbf{A}$, if and only if it is independent of I_4, I_5. In other words any orthogonal invariant $F(\mathbf{A})$ of the tensor $\mathbf{A}$ is a function of the form $F(\mathbf{A}) = f(\operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^2, \operatorname{tr}\mathbf{A}^3)$. Now we have to show that two symmetric tensors with the same eigenvalues are connected by means of transformation of pure rotation. Let us write down the theorem on spectral decomposition

$$\mathbf{A} = \sum A_i\, \mathbf{d}_i \otimes \mathbf{d}_i, \quad \mathbf{A}^* = \sum A_k\, \mathbf{d}_k^* \otimes \mathbf{d}_k^*,$$

where the triples of vectors $\mathbf{d}_i$ and $\mathbf{d}_k^*$ are orthonormal but they may have different orientation. Thus we may write down

$$\mathbf{d}_m^* = \mathbf{Q}\cdot\mathbf{d}_m = \mathbf{d}_m\cdot\mathbf{Q}^T \quad \Rightarrow \quad \mathbf{A}^* = \mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T,$$

where $\mathbf{Q}$ is an orthogonal tensor. If $\det \mathbf{Q} = 1$, then the ***I-problem*** has been solved. If $\det \mathbf{Q} = -1$, then $\mathbf{Q}$ may be represented as decomposition $\mathbf{Q} = \mathbf{P}\cdot(-\mathbf{E})$. Using this decomposition we have

$$\mathbf{A}^* = \mathbf{P}\cdot(-\mathbf{E})\cdot\mathbf{A}\cdot(-\mathbf{E})^T\cdot\mathbf{P}^T = \mathbf{P}\cdot\mathbf{A}\cdot\mathbf{P}^T, \quad \det \mathbf{P} = 1,$$

and the ***I-problem*** has been solved as well.

10 Basic invariants of a collection of vector and of tensor

Let us consider a collection of vector $\mathbf{a}$ and of symmetric second-rank tensor $\mathbf{A}$. In literature [6] it is supposed that the set of six invariants

$$I_1 = \operatorname{tr}\mathbf{A}, \quad I_2 = \operatorname{tr}\mathbf{A}^2, \quad I_3 = \operatorname{tr}\mathbf{A}^3, \quad I_4 = \mathbf{a}\cdot\mathbf{a}, \quad I_5 = \mathbf{a}\cdot\mathbf{A}\cdot\mathbf{a}, \quad I_6 = \mathbf{a}\cdot\mathbf{A}^2\cdot\mathbf{a} \tag{37}$$

fixes this collection.

Almost the same case we have for a collection of symmetric second-rank tensor $\mathbf{A}$ and of skew-symmetric second-rank tensor $\mathbf{W}$. It is supposed [6] that for this case it is necessary to set seven integrals

$$I_1 = \operatorname{tr}\mathbf{A}, \quad I_2 = \operatorname{tr}\mathbf{A}^2, \quad I_3 = \operatorname{tr}\mathbf{A}^3,$$

$$I_4' = \operatorname{tr}\mathbf{W}^2, \quad I_5' = \operatorname{tr}\left(\mathbf{W}^2\cdot\mathbf{A}\right), \quad I_6' = \operatorname{tr}\left(\mathbf{W}^2\cdot\mathbf{A}^2\right), \quad I_7' = \operatorname{tr}\left(\mathbf{W}^2\cdot\mathbf{A}\cdot\mathbf{W}\cdot\mathbf{A}^2\right). \tag{38}$$

From the other hand it is well known that for any skew-symmetric tensor $\mathbf{W}$ there exists uniquely defined vector $\mathbf{a}$ such that

$$\mathbf{W} = \mathbf{a}\times\mathbf{E}.$$

Making use of this representation one may obtain

$$I_4' = -2\mathbf{a}\cdot\mathbf{a}, \quad I_5' = \mathbf{a}\cdot\mathbf{A}\cdot\mathbf{a} - a^2 \mathrm{tr}\mathbf{A}, \quad I_6' = \mathbf{a}\cdot\mathbf{A}^2\cdot\mathbf{a} - a^2\mathrm{tr}\mathbf{A}^2, \quad I_7' = -\mathbf{a}\cdot\mathbf{A}^2\cdot(\mathbf{a}\times\mathbf{A}\cdot\mathbf{a})\,. \tag{39}$$

By this reason it seems to be clear that the lists of invariants (37) and (38) must be the same. But it is not so. One may think that the difference is arising due to following fact. The vector $\mathbf{a}$ in (37) is a polar one, but the vector $\mathbf{a}$ in (39) is an axial one. However it can not be the reason since the dimension of invariant basis is independent of the tensor kind.

Let us apply our approach to this case. For visual perception it is useful to keep in mind that a collection of vector $\mathbf{a}$ and symmetric second-rank tensor $\mathbf{A}$ is equivalent to the collection

$$\mathbf{A}, \quad \mathbf{a}, \quad \mathbf{A}\cdot\mathbf{a}, \quad \mathbf{A}^2\cdot\mathbf{a}.$$

Therefore it is necessary to find the list of invariants, whose fixation determines this collection as a rigid whole. For a collection of vector $\mathbf{a}$ and symmetric second-rank tensor $\mathbf{A}$ the basic equation (22) takes the form

$$\frac{\partial F}{\partial \mathbf{a}}\cdot(\mathbf{m}\times\mathbf{a}) + \left(\frac{\partial F}{\partial \mathbf{A}}\right)^T\cdot\cdot(\mathbf{m}\times\mathbf{A} - \mathbf{A}\times\mathbf{m}) = 0. \tag{40}$$

The characteristic system for (40)

$$\frac{d\mathbf{a}}{ds} = \mathbf{m}\times\mathbf{a}, \quad \frac{d\mathbf{A}}{ds} = \mathbf{m}\times\mathbf{A} - \mathbf{A}\times\mathbf{m}.$$

This system of ninth order has exactly eight independent integrals. However only six from them are independent of arbitrary vector $\mathbf{m}$. These integrals are given by expressions

$$I_1 = \mathrm{tr}\,\mathbf{A}, \quad I_2 = \mathrm{tr}\,\mathbf{A}^2, \quad I_3 = \mathrm{tr}\,\mathbf{A}^3, \quad I_4 = \mathbf{a}\cdot\mathbf{a}, \quad I_5 = \mathbf{a}\cdot\mathbf{A}\cdot\mathbf{a},$$
$$I_6 = \mathbf{a}\cdot\mathbf{A}^2\cdot\mathbf{a}, \quad I_7 = \mathbf{a}\cdot\mathbf{A}^2\cdot(\mathbf{a}\times\mathbf{A}\cdot\mathbf{a})\,. \tag{41}$$

The list (41) contain seven integrals but between them there is a relation

$$I_7^2 = \begin{vmatrix} I_4 & I_5 & I_6 \\ I_5 & I_6 & \mathbf{a}\cdot\mathbf{A}^3\cdot\mathbf{a} \\ I_6 & \mathbf{a}\cdot\mathbf{A}^3\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{A}^4\cdot\mathbf{a} \end{vmatrix}, \tag{42}$$

which was not mentioned in literature. The invariants $\mathbf{a}\cdot\mathbf{A}^3\cdot\mathbf{a}$ and $\mathbf{a}\cdot\mathbf{A}^4\cdot\mathbf{a}$ in the determinant (42) must be expressed in terms of invariants I_1 – I_6 with the help of the Cayley–Hamilton identity.

The invariant I_7 must be taken into account and can not be ignored. In order to verify this fact it is enough to consider two collections

$$\mathbf{A}, \quad \mathbf{a} \quad \text{and} \quad \mathbf{B} = \mathbf{S}_n\cdot\mathbf{A}\cdot\mathbf{S}_n^T, \quad \mathbf{a},$$

where $\mathbf{S}_n = \mathbf{E} - 2\mathbf{n}\otimes\mathbf{n}$, $\mathbf{n}\cdot\mathbf{n} = 1$, $\mathbf{n}\cdot\mathbf{a} = 0$. It is easy to check that invariants I_1 – I_6 for these two sets are the same. However we have

$$\mathbf{a}\cdot\mathbf{B}^2\cdot(\mathbf{a}\times\mathbf{B}\cdot\mathbf{a}) = -\mathbf{a}\cdot\mathbf{A}^2\cdot(\mathbf{a}\times\mathbf{A}\cdot\mathbf{a}),$$

i.e. the triples of vectors $\mathbf{a}$, $\mathbf{A}\cdot\mathbf{a}$, $\mathbf{A}^2\cdot\mathbf{a}$ and $\mathbf{a}$, $\mathbf{B}\cdot\mathbf{a}$, $\mathbf{B}^2\cdot\mathbf{a}$ have different orientations and can not be combined by a rotation.

Finally we obtain that the collection of vector $\mathbf{a}$ and symmetric second-rank tensor $\mathbf{A}$ has exactly six functionally independent invariants (41)–(42). Now we have to show that the fixation of the invariants (41)–(42) determines the collection of vector $\mathbf{a}$ and symmetric second-rank tensor $\mathbf{A}$ as a rigid whole. Let us consider two sets $\mathbf{a}$, $\mathbf{A}$ and $\mathbf{b}$, $\mathbf{B}$. If invariants $I_1 - I_4$ for these sets are the same, then we have

$$\mathbf{a} = \mathbf{Q}_a\cdot\mathbf{b}, \quad \mathbf{A} = \mathbf{Q}_A\cdot\mathbf{B}\cdot\mathbf{Q}_A^T, \tag{43}$$

where $\mathbf{Q}_a$ and $\mathbf{Q}_A$ are any orthogonal tensors. Tensors $\mathbf{Q}_a$ and $\mathbf{Q}_A$ must ensure the coincidence of invariants $I_5 - I_7$ for these two sets. Thus we have

$$\mathbf{a}\cdot\mathbf{A}\cdot\mathbf{a} \equiv \mathbf{b}\cdot\mathbf{Q}\cdot\mathbf{B}\cdot\mathbf{Q}^T\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{B}\cdot\mathbf{b}, \quad \mathbf{a}\cdot\mathbf{A}^2\cdot\mathbf{a} \equiv \mathbf{b}\cdot\mathbf{Q}\cdot\mathbf{B}^2\cdot\mathbf{Q}^T\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{B}^2\cdot\mathbf{b},$$

where

$$\mathbf{Q} \equiv \mathbf{Q}_a^T\cdot\mathbf{Q}_A.$$

From these equation it follows

$$\mathbf{Q} = (\det\mathbf{S}_b)^\alpha(\det\mathbf{S}_B)^\beta\,\mathbf{S}_b\cdot\mathbf{S}_B,$$

where $\mathbf{S}_b$ and $\mathbf{S}_B$ are some elements of symmetry of vector $\mathbf{b}$ and of tensor $\mathbf{B}$ correspondingly, $\alpha = 0$ for polar $\mathbf{b}$, $\alpha = 1$ for axial $\mathbf{b}$, $\beta = 0$ for polar $\mathbf{B}$, $\beta = 1$ for axial $\mathbf{B}$. Now we have

$$\mathbf{Q}_A = (\det\mathbf{S}_b)^\alpha(\det\mathbf{S}_B)^\beta\,\mathbf{Q}_a\cdot\mathbf{S}_b\cdot\mathbf{S}_B,$$

The relation (43) takes the form

$$\mathbf{a} = \mathbf{Q}_a\cdot\mathbf{b}, \quad \mathbf{A} = \mathbf{Q}_a\cdot\mathbf{S}_b\cdot\mathbf{B}\cdot\mathbf{S}_b^T\cdot\mathbf{Q}_a^T,$$

or

$$\mathbf{a} = \mathbf{P}\cdot\mathbf{b}, \quad \mathbf{A} = \mathbf{P}\cdot\mathbf{B}\cdot\mathbf{P}^T, \quad \mathbf{P} \equiv (\det\mathbf{S}_b)^\alpha\mathbf{Q}_a\cdot\mathbf{S}_b \tag{44}$$

Taking into account (44) and the expression for I_7 one may write down

$$\mathbf{a}\cdot\mathbf{A}^2\cdot(\mathbf{a}\times\mathbf{A}\cdot\mathbf{a}) = (\det\mathbf{P})\,\mathbf{b}\cdot\mathbf{B}^2\cdot(\mathbf{b}\times\mathbf{B}\cdot\mathbf{b}) \tag{45}$$

Since the $I_7(\mathbf{a}, \mathbf{A})$ must be equal to $I_7(\mathbf{b}, \mathbf{B})$ then we obtain $\det\mathbf{P} = 1$, i.e. the tensor $\mathbf{P}$ must be the proper orthogonal tensor.

11 Basic invariants for a set of two tensors

Let us consider a collection of two symmetric tensors

$$\mathbf{A} = \sum_{k=1}^{3} A_k\,\mathbf{a}_k\otimes\mathbf{a}_k, \quad \mathbf{B} = \sum_{k=1}^{3} B_k\,\mathbf{b}_k\otimes\mathbf{b}_k. \tag{46}$$

It is claimed that in order to fix this system it is necessary to fix the next ten invariants

$$I_k^A = \operatorname{tr}\mathbf{A}^k, \quad I_k^B = \operatorname{tr}\mathbf{B}^k, \quad x = \operatorname{tr}(\mathbf{A}\cdot\mathbf{B}),$$

$$y = \mathrm{tr}\,(\mathbf{A}^2 \cdot \mathbf{B}), \quad z = \mathrm{tr}\,(\mathbf{A} \cdot \mathbf{B}^2), \quad u = \mathrm{tr}\,(\mathbf{A}^2 \cdot \mathbf{B}^2). \tag{47}$$

In the literature the tensors $\mathbf{A}$, $\mathbf{B}$ are supposed to be polar. Let us note that for application the case, when $\mathbf{A}$ is a polar tensor and $\mathbf{B}$ is an axial tensor, is much more important. Accordingly to the theorem on the dimension of the invariant basis, the number of functionally independent invariants is equal to nine. Therefore the invariants shown in the list (47) can not be functionally independent. That means that there exists some relation superposed on the invariants (47).

Let us show that the invariants x, y, z, u in (47) can be expressed through the invariants A_k, B_n and four parameters on which one obvious relation is superposed. For sake of simplicity the eigenvalues A_k and B_n are supposed to be different. In such a case we may write

$$\mathbf{a}_k \otimes \mathbf{a}_k = \alpha_k \mathbf{\Gamma}_k(\mathbf{A}), \ \alpha_k^{-1} = (A_k - A_i)(A_k - A_j), \tag{48}$$

$$\mathbf{\Gamma}_k(\mathbf{A}) \equiv \mathbf{A}^2 - (I_1^A - A_k)\mathbf{A} + A_k^{-1} I_3^A \mathbf{E}, \quad i \neq j \neq k \neq i;$$

$$\mathbf{b}_m \otimes \mathbf{b}_m = \beta_m \mathbf{\Gamma}_m(\mathbf{B}), \quad \beta_m^{-1} = (B_m - B_n)(B_m - B_p),$$

$$\mathbf{\Gamma}_m(\mathbf{B}) \equiv \mathbf{B}^2 - (I_1^B - B_m)\mathbf{B} + B_m^{-1} I_3^B \mathbf{E}, \ m \neq n \neq p \neq m.$$

Making use of (48) and (11) one may obtain

$$(\mathbf{a}_k \cdot \mathbf{b}_m)^2 = \alpha_k \beta_m \left[u + (A_k - I_1^A)(B_m - I_1^B)x - (I_1^B - B_m)y - (I_1^A - A_k)z + \right.$$
$$\left. + \left(B_m I_1^B - 2J_B\right) A_k^{-1} I_3^A + \left(A_k I_1^A - 2J_A\right) B_m^{-1} I_3^B + 3\,(A_k B_m)^{-1} I_3^A I_3^B \right], \tag{49}$$

$$2J_A = (\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2), \quad 2J_B = (\mathrm{tr}\mathbf{B})^2 - \mathrm{tr}(\mathbf{B}^2).$$

The system (49) of nine equations is the linear system for four invariants x, y, z and u. It may be verified that the rank of the system (49) is equal to four. Therefore the invariants x, y, z and u may be expressed as function of eigenvalues A_k, B_n and and numbers $\mathbf{a}_k \cdot \mathbf{b}_m$, which are connected by six constraints

$$\sum_{m=1}^{3} (\mathbf{a}_k \cdot \mathbf{b}_m)^2 = 1, \ \sum_{k=1}^{3} (\mathbf{a}_k \cdot \mathbf{b}_m)^2 = 1.$$

Let the triples $\mathbf{a}_k$ and $\mathbf{a}_k$ have the same orientation. This assumption does not restrict the analysis. In such a case we may write down

$$\mathbf{b}_k = \mathbf{Q} \cdot \mathbf{a}_k, \quad det\mathbf{Q} = 1. \tag{50}$$

The turn-tensor $\mathbf{Q}$ may be expressed in terms of turn-vector $\boldsymbol{\theta}$

$$\mathbf{Q}(\boldsymbol{\theta}) = \cos\theta\,\mathbf{E} + \frac{(1 - \cos\theta)}{\theta^2} \boldsymbol{\theta} \otimes \boldsymbol{\theta} + \frac{\sin\theta}{\theta}\,\boldsymbol{\theta} \times \mathbf{E}, \tag{51}$$

where $\theta = |\boldsymbol{\theta}|$.

Making use of (51) and (50), one may obtain

$$\theta^2 \mathbf{a}_p \cdot \mathbf{b}_p = \theta_p^2 + \cos\theta\,(\theta^2 - \theta_p^2), \quad \theta^2 = \theta_1^2 + \theta_2^2 + \theta_3^2, \tag{52}$$

$$(\mathbf{a}_m \cdot \mathbf{b}_p)^2 = \frac{(1-\cos\theta)^2}{\theta^4}\theta_m^2\theta_p^2 + \frac{\sin^2\theta}{\theta^2}\theta_s^2 + \\ + e_{mps}\frac{2\sin\theta(1-\cos\theta)}{\theta^3}\theta_1\theta_2\theta_3, \quad m \neq p \neq s \neq m, \quad (53)$$

where e_{mks} is the permutation symbol, $\theta_k = \boldsymbol{\theta} \cdot \mathbf{a}_k$.

Let us show that invariants x, y, z, u can be expressed in term of the turn-vector components and eigenvalues of the tensor $\mathbf{A}$ and $\mathbf{B}$. For this end in the right side of (49) we may ignore the terms depending on the eigenvalues of the tensor $\mathbf{A}$ and $\mathbf{B}$ only and rewrite (49) in the form of three independent systems

$$u_1 - (B_2 + B_3)y_1 = a_{11}, \quad u_1 - (B_1 + B_3)y_1 = a_{12}, \quad u_1 - (B_1 + B_2)y_1 = a_{13}; \quad (54)$$

$$u_2 - (B_2 + B_3)y_2 = a_{21}, \quad u_2 - (B_1 + B_3)y_2 = a_{22}, \quad u_2 - (B_1 + B_2)y_2 = a_{23}; \quad (55)$$

$$u_3 - (B_2 + B_3)y_3 = a_{31}, \quad u_3 - (B_1 + B_3)y_3 = a_{32}, \quad u_3 - (B_1 + B_2)y_3 = a_{33}. \quad (56)$$

In (54)–(56) the next new unknown variables are introduced

$$u_m = u - (A_n + A_p)z, \quad y_m = y - (A_n + A_p)x, \quad (57)$$

where $m \neq n \neq p \neq m$. Besides the notation are used

$$a_{ik} = \mathbf{a}_i \cdot \mathbf{b}_k / \alpha_i \beta_k.$$

Solutions of (54)–(56) are given by expressions

$$u_m = a_{mm} + \frac{B_n + B_p}{B_n - B_p}(a_{mn} - a_{mp}), \quad y_m = \frac{a_{mn} - a_{mp}}{B_n - B_p}, \quad (58)$$

where the numbers m, n, p consist the even permutation of 1, 2, 3. It is easy to find the solution of (57)

$$x = \frac{y_1 - y_2}{A_1 - A_2}, \quad y = y_3 + \frac{A_1 + A_2}{A_1 - A_2}(y_1 - y_2),$$
$$z = \frac{u_1 - u_2}{A_1 - A_2}, \quad u = u_3 + \frac{A_1 + A_2}{A_1 - A_2}(u_1 - u_2).$$

Making use of (58) we finally obtain

$$x = (A_1 - A_3)\left[(B_2 - B_1)(\mathbf{a}_1 \cdot \mathbf{b}_2)^2 + (B_3 - B_1)(\mathbf{a}_1 \cdot \mathbf{b}_3)^2\right].$$

The analogous formulae may be obtained for invariants y, z and u. It may be noted that invariants x, y, z, u are represented in terms of four numbers θ_1^2, θ_2^2, θ_3^2, $\theta_1\theta_2\theta_3$. However there is one obvious constraint $\theta_1^2\theta_2^2\theta_3^2 = (\theta_1\theta_2\theta_3)^2$. Thus we see that the number of functionally independent invariants is equal to nine. This is the statement of the theorem on the dimension of the invariant basis.

The using of the turn-vector is possible but it is not convenient. So it would be better to use more simple invariants with clear physical sense. To this end let us introduce the tensors

$$\mathbf{C}(\mathbf{A}, \mathbf{B}) = \sum_{k=1}^{3} B_k \mathbf{b}_k \times \mathbf{A} \times \mathbf{b}_k = \sum_{k=1}^{3} A_k \mathbf{a}_k \times \mathbf{B} \times \mathbf{a}_k = \mathbf{C}(\mathbf{B}, \mathbf{A})$$

and

$$\mathbf{D}(\mathbf{A},\, \mathbf{B}) = \mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A} = -\mathbf{D}(\mathbf{B},\, \mathbf{A}).$$

The skew-symmetric tensor $\mathbf{D}(\mathbf{A}, \mathbf{B})$ may be represented as

$$\mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A} = -\mathbf{r}\times\mathbf{E} \quad \Rightarrow \quad \mathbf{r} = (\mathbf{A}\cdot\mathbf{B})_\times. \tag{59}$$

The vector $\mathbf{r}$ characterizes a non-coaxiality of the tensors $\mathbf{A}$ and $\mathbf{B}$. The tensor $\mathbf{C}$ is the isotropic function of the tensors $\mathbf{A}$ and $\mathbf{B}$. It may be represented a combination of the form

$$\mathbf{C} = [\mathrm{tr}(\mathbf{A}\cdot\mathbf{B}) - \mathrm{tr}\mathbf{A}\,\mathrm{tr}\mathbf{B}]\,\mathbf{E} + (\mathrm{tr}\mathbf{B})\,\mathbf{A} + (\mathrm{tr}\mathbf{A})\,\mathbf{B} - \mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A}.$$

The tensor $\mathbf{C}$ is convenient for applications since its eigenvalues characterize the reciprocal position of the tensors $\mathbf{A}$ and $\mathbf{B}$.

In order to fix the collection (46) it is possible to set the next nine invariants. Firstly, they contain the principal invariants of the tensors $\mathbf{A}$ and $\mathbf{B}$

$$I_1^A = \mathrm{tr}\,\mathbf{A}, \quad I_2^A = \frac{1}{2}\left[(\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}\,\mathbf{A}^2)\right], \quad I_3^A = \det\mathbf{A},$$

$$I_1^B = \mathrm{tr}\,\mathbf{B}, \quad I_2^B = \frac{1}{2}\left[(\mathrm{tr}\mathbf{B})^2 - \mathrm{tr}\,\mathbf{B}^2)\right], \quad I_3^B = \det\mathbf{B}. \tag{60}$$

Invariants (60) determine the eigenvalues A_k and B_k of the tensors $\mathbf{A}$ and $\mathbf{B}$. Secondly, in order to fix the reciprocal position of the tensors $\mathbf{A}$ and $\mathbf{B}$ it is possible to use the next three invariants

$$\mathrm{tr}\mathbf{C}, \quad \mathrm{tr}\mathbf{C}^2, \quad \mathbf{r}\cdot\mathbf{r}.$$

These invariants may be expressed in terms of invariants (47)

$$\mathrm{tr}\mathbf{C} = \mathrm{tr}(\mathbf{A}\cdot\mathbf{B}) - \mathrm{tr}\mathbf{A}\mathrm{tr}\mathbf{B}, \quad \mathbf{r}\cdot\mathbf{r} = \mathrm{tr}(\mathbf{A}^2\cdot\mathbf{B}^2) - \mathrm{tr}\left[(\mathbf{A}\cdot\mathbf{B})^2\right],$$

$$\mathrm{tr}\mathbf{C}^2 = [\mathrm{tr}(\mathbf{A}\cdot\mathbf{B})]^2 + \mathrm{tr}\mathbf{A}^2\mathrm{tr}\mathbf{B}^2 - 2\mathrm{tr}(\mathbf{A}^2\cdot\mathbf{B}^2).$$

Thus we may fix the next three invariants

$$\mathrm{tr}(\mathbf{A}\cdot\mathbf{B}), \quad \mathrm{tr}(\mathbf{A}^2\cdot\mathbf{B}^2), \quad \mathrm{tr}(\mathbf{A}^2\cdot\mathbf{B}^2) - \mathrm{tr}\left[(\mathbf{A}\cdot\mathbf{B})^2\right]. \tag{61}$$

Now we have to show that fixation of the nine invariants (60), (61) determines the set of the tensors $\mathbf{A}$ and $\mathbf{B}$ as a rigid whole. Let there be given two sets of the tensors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{A}_*$, $\mathbf{B}_*$. A fixation of the invariants (60) leads to the relations

$$\mathbf{A}_* = \mathbf{Q}_A\cdot\mathbf{A}\cdot\mathbf{Q}_A^T, \quad \mathbf{B}_* = \mathbf{Q}_B\cdot\mathbf{B}\cdot\mathbf{Q}_B^T.$$

where $\mathbf{Q}_A$ and $\mathbf{Q}_B$ are orthogonal tensors.

Fixation of the invariants (61) gives the next system of equation

$$\mathbf{A}\cdot\cdot\left(\mathbf{Q}\cdot\mathbf{B}\cdot\mathbf{Q}^T - \mathbf{B}\right) = 0, \quad \mathbf{A}^2\cdot\cdot\left(\mathbf{Q}\cdot\mathbf{B}^2\cdot\mathbf{Q}^T - \mathbf{B}^2\right) = 0,$$

$$\mathrm{tr}\left[\left(\mathbf{A}\cdot\mathbf{Q}\cdot\mathbf{B}\cdot\mathbf{Q}^T\right)^2\right] = \mathrm{tr}\left[(\mathbf{A}\cdot\mathbf{B})^2\right], \tag{62}$$

where

$$\mathbf{Q} \equiv \mathbf{Q}_A^T \cdot \mathbf{Q}_B.$$

From (62) it follows

$$\mathbf{Q} = \mathbf{S}_A \cdot \mathbf{S}_B \quad \Rightarrow \quad \mathbf{Q}_B = \mathbf{Q}_A \cdot \mathbf{S}_A \cdot \mathbf{S}_B,$$

where $\mathbf{S}_A$ and $\mathbf{S}_B$ are some elements of the tensors $\mathbf{A}$ and $\mathbf{B}$ correspondingly. Thus we have

$$\mathbf{A}_* = \mathbf{Q}_A \cdot \mathbf{S}_A \cdot \mathbf{A} \cdot \mathbf{S}_A^T \cdot \mathbf{Q}_A^T, \quad \mathbf{B}_* = \mathbf{Q}_A \cdot \mathbf{S}_A \cdot \mathbf{B} \cdot \mathbf{S}_A^T \cdot \mathbf{Q}_A^T.$$

For any $\mathbf{Q}_A$ we may chose such element of symmetry $\mathbf{S}_A$ that

$$\det(\mathbf{Q}_A \cdot \mathbf{S}_A) = 1.$$

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Nonlinear Theory of Thin Rods*

Abstract

Nonlinear theory of the rods is the oldest and, maybe, the most important theory in continuum mechanics of solids. However there are some theoretical problems which have no solution up to now. The report is devoted to discussion of the dynamic theory of the thin spatially bent and naturally twisted rods. The suggested theory includes all known variants of the theory of rods, but possesses wider branch of applicability. A new method of construction of the elasticity tensors is offered and their structure is established. To this end a new theory of the tensor symmetry in space with two independent orientations is essentially used. For plane elastic curves all modules of elasticity are determined. The significant attention in the report is given to the analysis of some classical problems, including those from them, solution of which leads to paradoxical results. In particular, it is in detail considered well-known elastica by Euler and it is shown, that alongside with known equilibrium configurations there are also dynamic equilibrium configurations. In this case the form of an elastic curve does not vary, but the bent rod makes rotations around of a vertical axis. Energy of deformation in this case does not vary. Let us note that these movements are not movements of a rod as the rigid whole for the clamped end face of the rod remains motionless. From this it follows, that the bent equilibrium configuration in the Euler elastica is, in contrast to the conventional point of view, unstable. On the other hand, this conclusion is not confirmed by experimental data. Therefore there is a paradoxical situation which demands the decision. The similar situation known under the name of the Nikolai paradox arises at torsion of a rod by the boundary twisting moment. In this case experiment shows that twisting moment produces stabilizing effect that is in the sharp contradiction with the theoretical data. In the report it is shown what to avoid the specified paradoxes it is possible at a special choice of the constitutive equation for the moment. It appears that the moment should depend in the special form on angular velocity. Last dependence is not connected with the presence (or absence) internal friction in the rod.

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1 The rod theory and modern mechanics

The theory of thin rods has played outstanding role in the history of development of mechanics and mathematical physics. In order to show the contribution of the theory of thin rods to the development of natural sciences more clearly, let us point out only some facts.

Birth of the ordinary differential equations. In 1691 Jacob Bernoulli has derived the differential equation of equilibrium of a rope (string)

$$\mathbf{N}' + \rho\mathbf{F} = \mathbf{0}. \tag{1}$$

The equation (1) was the first differential equation in the history of a science.

Birth of the equations in partial derivatives. In 1742 Jacque D'Alembert has derived the equation of vibrations of a string

$$\frac{\partial^2 u}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial^2 t^2} = 0. \tag{2}$$

The equation (2) was the first differential equation in partial derivatives. Development of the methods of its solution has led to the creation of ***the theory of decomposition of functions in series*** — Daniel Bernoulli and Leonard Euler.

Birth of the theory of bifurcation of the solutions of nonlinear differential equations. In 1744 L. Euler has solved a problem on a longitudinal bending of the rod, named later ***Euler's Elastica,*** and found the beginning of the theory of bifurcations and the theory of the eigenvalues of nonlinear operators.

Birth of a new mechanics and the proof of incompleteness of the Newton mechanics. In 1771 L. Euler has derived a general equations of equilibrium of rods

$$\mathbf{N}' + \rho\mathbf{F} = \mathbf{0}, \quad \mathbf{M}' + \mathbf{R}' \times \mathbf{N} + \rho\mathbf{L} = \mathbf{0}. \tag{3}$$

To derive the equations (3) it was required to Euler about 50 years of reflections. As a result Euler has made one of the greatest opening in mechanics and physics, which to the full extent is not realized by the majority of mechanics and physicists up to present time. Namely, Euler has realized the necessity of the introduction of moments as independent objects, which can not be in terms of the moment of force. That means, firstly, necessity of the introductions of the new fundamental law of physics, expressed by the second equation (3) and, secondly, the fundamental incompleteness of the Newton Mechanics. Though L. Euler has made the determining step for introduction of the moments, independent of forces, but the general definition of the moment has been given rather recently by P.A. Zhilin.

Birth of the theory of stability of the nonconservative systems. In 1927 E.L. Nikolai has reported the results of the analysis of stability of the equilibrium configuration of the rod under the action of the twisting moment. He has shown, that this configuration is unstable at any as much as small value of the twisting moment (the Nikolai Paradox). The scientists of that time were shocked by this result for it was in sharp contrast with the conventional Euler's concept of critical forces. Then P.F. Papkovitch has specified, that the Nikolai problem deals with the nonconservative system. Therefore it is not necessary to be surprised to the obtained result because it is possible of accumulation of the energy in system. The subsequent development of the theory of stability of nonconservative systems has revealed also others surprising facts, for example, destabilizing role of internal friction. In the report it will be shown, that paradox of Nikolai is due to reasons which has no direct relation with the nonconservativeness of system. Nevertheless, the theory of stability of nonconservative systems now is one of the important branches of mechanics.

Birth of the symmetry theory in multi-oriented spaces. In 1977 P.A. Zhilin at construction of the constitutive equations in the theory of rods and shells has found out, that the application of the classical theory of symmetry leads to the absurd results. The analysis has shown, that the reason of the impasse is that fact, that the theory of rods and shells contain tensor's objects that is defined in spaces with two independent orientations. Therefore in such space there exist the tensors of four various types. The classical theory of symmetry is applicable only to the so-called polar tensors, i.e. to objects, independent of a choice of orientations in space. Thus it was necessary to develop the generalized theory of symmetry, which is valid for tensors of any types. Let us note that without this generalized theory of symmetry the correct construction of a general theory of rods and shells is impossible.

Above only those facts have been marked which have affected and continue to influence on the theoretical foundations of modern mechanics and mathematical physics. In the report there is no need to speak about enormous value of the rod theory for decision of actual problems of technics. Unfortunately, frameworks of the report do not allow to tell about remarkable achievements of many researchers at the decision of the very much interesting specific problems.

Unsolved questions of the rod theory. In the rod theory it is obtained a lot of surprising and even paradoxical results which demand clear explanations. Spatial forms of the rod motions are not almost investigated. Within the framework of the existing theory of rods it is very difficult to investigate the important problems for related dynamics of rods and, for example, rigid bodies as these two two important objects of mechanics are stated on various and incompatible languages. The main obstacle in a way of overcoming of all these difficulties is absence a general nonlinear theory of rods stated in language convenient for applications. The first presentation of such theory is one of the purposes of the report. Another, not less important, the purpose of the report is the discussion, from positions of the submitted theory, of some classical problems of the rod theory and revealing in them of the new circumstances latent in existing decisions. In particular, the new interpretation of the Nikolai paradox based on the full analysis of the Euler elastica will be given in the report. The author has solutions of a several new problems, but, unfortunately, is forced to leave them behind frameworks of the report.

2 The model of rod

The model of thin rod is the directed curve, which is defined by fixation of the vector $\mathbf{r}(s)$ and triple $\mathbf{d}_m$

$$\{\mathbf{r}(s),\ \mathbf{d}_1(s),\ \mathbf{d}_2(s),\ \mathbf{d}_3(s)\},\quad \mathbf{d}_m\cdot\mathbf{d}_n=\delta_{mn},\quad 0\le s\le l, \tag{4}$$

where s is the length of the curve arc, l — the length of curve. The vector $\mathbf{r}(s)$ in (4) determines the carrying curve with natural triple $\{\mathbf{t}_1\equiv\mathbf{t},\ \mathbf{t}_2\equiv\mathbf{n},\ \mathbf{t}_3\equiv\mathbf{b}=\mathbf{t}\times\mathbf{n}\}$, where the vectors **t**, **n** and **b** are unit vectors of the tangent, normal and binormal respectively. For natural triple one has

$$\mathbf{t}_i' = \boldsymbol{\tau} \times \mathbf{t}_i, \quad \boldsymbol{\tau}(s) = R_t^{-1}(s)\mathbf{t}(s) - R_c^{-1}(s)\mathbf{b}(s), \tag{5}$$

where R_c is the radius of curvature and R_t is the radius of twisting, $\boldsymbol{\tau}$ is the Darboux vector. Thus, in each point of the curve the two triples are given: natural triple $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and additional triple $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3 = \mathbf{t}\}$. The vectors $(\mathbf{n}, \mathbf{b})$ and $(\mathbf{d}_1, \mathbf{d}_2)$ are placed in the cross plane to the undeformed curve and determines the cross-section of the undeformed rod, but, in general, does not coincide. In what follows the vectors $(\mathbf{d}_1, \mathbf{d}_2)$ are principal axes of inertia of the cross-section. The changing of the triple $\mathbf{d}_k(s)$ under the motion along the curve is determined by the vector $\mathbf{q}(s)$ such that

$$\mathbf{d}_k'(s) = \mathbf{q}(s) \times \mathbf{d}_k(s). \tag{6}$$

It is easy to find the relation between $\mathbf{q}$ and $\boldsymbol{\tau}$

$$\mathbf{q} = \left(\varphi' + R_t^{-1}\right)\mathbf{t} - R_c^{-1}\mathbf{b} = \varphi'\mathbf{t} + \boldsymbol{\tau}, \tag{7}$$

where φ is called the angle the natural twisting of the rod.

The motion of the rod is defined by

$$\mathbf{r}(s) \quad \to \quad \mathbf{R}(s,t); \qquad \mathbf{d}_k(s) \quad \to \quad \mathbf{D}_k(s,t)$$

or

$$\mathbf{R}(s,t) = \mathbf{r}(s) + \mathbf{u}(s,t), \qquad \mathbf{D}_k(s,t) = \mathbf{P}(s,t) \cdot \mathbf{d}_k(s), \tag{8}$$

where $\mathbf{u}(s,t)$ is the displacement vector, $\mathbf{P}(s,t)$ is the turn-tensor. The translational velocity and angular velocity are defined by

$$\mathbf{V}(s,t) = \dot{R}(s,t), \quad \dot{\mathbf{P}}(s,t) = \boldsymbol{\omega}(s,t) \times \mathbf{P}(s,t), \quad \dot{f} \equiv df/dt. \tag{9}$$

If the turn-tensor $\mathbf{P}(s,t)$ is given, then

$$\boldsymbol{\omega}(s,t) = -\frac{1}{2}\left[\dot{\mathbf{P}} \cdot \mathbf{P}^T\right]_\times, \quad (\mathbf{a} \otimes \mathbf{b})_\times \equiv \mathbf{a} \times \mathbf{b}. \tag{10}$$

3 Fundamental laws of mechanics

The first and the second laws of dynamics by Euler have an almost conventional form

$$\mathbf{N}'(s,t) + \rho_0\mathcal{F}(s,t) = \rho_0\left(\mathbf{V} + \underline{\boldsymbol{\Theta}_1 \cdot \boldsymbol{\omega}}\right)^{\cdot}, \tag{11}$$

$$\mathbf{M}' + \mathbf{R}' \times \mathbf{N} + \rho_0\mathcal{L} = \underline{\rho_0\mathbf{V} \times \boldsymbol{\Theta}_1 \cdot \boldsymbol{\omega}} + \rho_0\left(\underline{\mathbf{V} \cdot \boldsymbol{\Theta}_1} + \boldsymbol{\Theta}_2 \cdot \boldsymbol{\omega}\right)^{\cdot}, \tag{12}$$

where the underlined terms had been never taken into account, for the curved rods they are important.

Let us write down the energy balance equation (George Green, 1839)

$$\rho_0\dot{\mathcal{U}} = \mathbf{N} \cdot (\mathbf{V}' + \mathbf{R}' \times \boldsymbol{\omega}) + \mathbf{M} \cdot \boldsymbol{\omega}' + h' + \rho_0\mathcal{Q}, \tag{13}$$

Let the vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$ be the vector of extension-shear deformation and the vector of bending-twisting deformation respectively. They are defined as

$$\boldsymbol{\mathcal{E}} = \mathbf{R}' - \mathbf{P} \cdot \mathbf{t}, \quad \mathbf{P}' = \boldsymbol{\Phi} \times \mathbf{P}. \tag{14}$$

The Cartan equation

$$\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}} = \mathbf{V}' + \mathbf{R}' \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\Phi}} - \boldsymbol{\omega} \times \boldsymbol{\Phi} = \boldsymbol{\omega}'. \tag{15}$$

Putting (15) into (13), we obtain the energy balance equation in the next form

$$\rho_0 \dot{\mathcal{U}} = \mathbf{N} \cdot \left(\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}}\right) + \mathbf{M} \cdot (\dot{\boldsymbol{\Phi}} - \boldsymbol{\omega} \times \boldsymbol{\Phi}) + h' + \rho_0 \mathcal{Q}, \tag{16}$$

4 Reduced equation of the balance equation

The force $\mathbf{N}$ and the moment $\mathbf{M}$ in the rod may be represented as superposition of the elastic ($\mathbf{N}_e$, $\mathbf{M}_e$) and dissipative ($\mathbf{N}_d$, $\mathbf{M}_d$) terms

$$\mathbf{N} = \mathbf{N}_e(\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \mathbf{P}) + \mathbf{N}_d(s,t), \quad \mathbf{M} = \mathbf{M}_e(\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \mathbf{P}) + \mathbf{M}_d(s,t).$$

Let the parameter $\vartheta(s,t)$ is the temperature of the rod measured by some thermometer. That means that the temperature is the experimentally measured parameter. Let us introduce a new function η called the entropy. Let us define this function by the equation

$$\vartheta \dot{\eta} = h' + \rho_0 \mathcal{Q} + \mathbf{N}_d \cdot \left(\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}}\right) + \mathbf{M}_d \cdot (\dot{\boldsymbol{\Phi}} - \boldsymbol{\omega} \times \boldsymbol{\Phi}) \tag{17}$$

Let us point out that such definition of the entropy does not need in distinction between reversible and irreversible processes. Introduction of the entropy by the equality (17) is possible for any processes. The equality (17) is called the equation of the heat conduction.

Making use of (17) the energy balance equation (16) may be represented in the form

$$\rho_0 \dot{\mathcal{U}} = \mathbf{N}_e \cdot \left(\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}}\right) + \mathbf{M}_e \cdot \left(\dot{\boldsymbol{\Phi}} - \boldsymbol{\omega} \times \boldsymbol{\Phi}\right) + \vartheta \dot{\eta}. \tag{18}$$

The equation (18) is called ***the reduced energy balance equation***. Let us suppose that

$$\mathcal{U} = \mathcal{U}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \mathbf{P}, \eta).$$

It is clear that that the internal energy does not change under the superposition of rigid motions. Let us consider the two motions: $\mathbf{R}(s,t)$, $\mathbf{P}(s,t)$ and $\mathbf{R}_*(s,t)$, $\mathbf{P}_*(s,t)$, which are related by the equality

$$\mathbf{R}_*(s,t) - \mathbf{R}_*(\tilde{s},t) = \mathbf{Q}(\alpha) \cdot [\mathbf{R}(s,t) - \mathbf{R}(\tilde{s},t)], \quad \mathbf{P}_*(s,t) = \mathbf{Q}(\alpha) \cdot \mathbf{P}(s,t),$$

where $\mathbf{Q}(\alpha)$ is the set of properly orthogonal tensors, s and $\tilde{s}$ are two any points of the rod. It is easy to find that

$$\boldsymbol{\mathcal{E}}_*(s,t) = \mathbf{R}'_* - \mathbf{P}_* \cdot \mathbf{t} = \mathbf{Q}(\alpha) \cdot \boldsymbol{\mathcal{E}}(s,t),$$

$$\mathbf{\Phi}_*(s,t) = -\frac{1}{2}\left[\mathbf{P}'_* \cdot \mathbf{P}^{\mathsf{T}}_*\right]_\times = -\frac{1}{2}\left[\mathbf{Q} \cdot \mathbf{P}'_* \cdot \mathbf{P}^{\mathsf{T}}_* \cdot \mathbf{Q}^{\mathsf{T}}\right]_\times = \mathbf{Q}(\alpha) \cdot \mathbf{\Phi}(s,t).$$

Thus the internal energy must satisfy the next equality

$$\mathcal{U}(\boldsymbol{\mathcal{E}}_*, \mathbf{\Phi}_*, \mathbf{P}_*, \eta) = \mathcal{U}[\mathbf{Q}(\alpha) \cdot \boldsymbol{\mathcal{E}},\ \mathbf{Q}(\alpha) \cdot \mathbf{\Phi},\ \mathbf{Q}(\alpha) \cdot \mathbf{P},\ \eta] = \mathcal{U}(\boldsymbol{\mathcal{E}}, \mathbf{\Phi}, \mathbf{P}, \eta). \tag{19}$$

For the tensor $\mathbf{Q}(\alpha)$ we may accept

$$\frac{d}{d\alpha}\mathbf{Q}(\alpha) = \boldsymbol{\zeta}(\alpha) \times \mathbf{Q}(\alpha), \quad \mathbf{Q}(0) = \mathbf{E}, \quad \boldsymbol{\zeta}(0) = \boldsymbol{\omega}(t).$$

Differentiating the equality (19) with respect to α and accepting $\alpha = 0$, we have the equation

$$-\left(\frac{\partial \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}} \times \boldsymbol{\mathcal{E}} + \frac{\partial \mathcal{U}}{\partial \mathbf{\Phi}} \times \mathbf{\Phi}\right) \cdot \boldsymbol{\omega} + \left(\frac{\partial \mathcal{U}}{\partial \mathbf{P}}\right)^{\mathsf{T}} \cdot\cdot\, (\boldsymbol{\omega} \times \mathbf{P}) = 0. \tag{20}$$

Besides we have

$$\frac{d\mathcal{U}}{dt} = \frac{\partial \mathcal{U}}{\partial \eta}\dot{\eta} + \frac{\partial \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}} \cdot \dot{\boldsymbol{\mathcal{E}}} + \frac{\partial \mathcal{U}}{\partial \mathbf{\Phi}} \cdot \dot{\mathbf{\Phi}} + \left(\frac{\partial \mathcal{U}}{\partial \mathbf{P}}\right)^{\mathsf{T}} \cdot\cdot\, (\boldsymbol{\omega} \times \mathbf{P}) = 0.$$

Taking into account the equality (20) this equation may be rewritten as

$$\frac{d\mathcal{U}}{dt} = \frac{\partial \mathcal{U}}{\partial \eta}\dot{\eta} + \frac{\partial \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}} \cdot \left(\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}}\right) + \frac{\partial \mathcal{U}}{\partial \mathbf{\Phi}} \cdot \left(\dot{\mathbf{\Phi}} - \boldsymbol{\omega} \times \mathbf{\Phi}\right).$$

Putting this equality into (18) we obtain

$$\left(\frac{\partial \rho_0 \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}} - \mathbf{N}_e\right) \cdot \left(\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}}\right) + \left(\frac{\partial \rho_0 \mathcal{U}}{\partial \mathbf{\Phi}} - \mathbf{M}_e\right) \cdot \left(\dot{\mathbf{\Phi}} - \boldsymbol{\omega} \times \mathbf{\Phi}\right) + \\ + \left(\frac{\partial \rho_0 \mathcal{U}}{\partial \eta} - \vartheta\right)\dot{\eta} = 0. \tag{21}$$

The equation (21) must be valid for any processes and for arbitrary values of the vectors $\dot{\boldsymbol{\mathcal{E}}} - \boldsymbol{\omega} \times \boldsymbol{\mathcal{E}}$ and $\dot{\mathbf{\Phi}} - \boldsymbol{\omega} \times \mathbf{\Phi}$. It is possible if and only if the Cauchy-Green formulas are valid

$$\mathbf{N}_e = \frac{\partial \rho_0 \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}}, \quad \mathbf{M}_e = \frac{\partial \rho_0 \mathcal{U}}{\partial \mathbf{\Phi}}, \quad \vartheta = \frac{\partial \rho_0 \mathcal{U}}{\partial \eta}. \tag{22}$$

Besides accepting in (19) $\mathbf{Q} = \mathbf{P}^{\mathsf{T}}$ we see that the intrinsic energy is the function of the next argument

$$\mathcal{U} = \mathcal{U}(\boldsymbol{\mathcal{E}}_\times,\ \mathbf{\Phi}_\times,\ \eta), \quad \boldsymbol{\mathcal{E}}_\times \equiv \mathbf{P}^{\mathsf{T}} \cdot \boldsymbol{\mathcal{E}}, \quad \mathbf{\Phi}_\times \equiv \mathbf{P}^{\mathsf{T}} \cdot \mathbf{\Phi}. \tag{23}$$

The vectors $\boldsymbol{\mathcal{E}}_\times$ and $\mathbf{\Phi}_\times$ are called the energetic vectors of deformation.

Axisymmetrical vibrations of a ring.

$$\mathbf{R}(s,t) = [a + w(t)]\mathbf{n}(s), \quad \mathbf{P}(s,t) = \mathbf{E} \quad \Rightarrow \quad \mathbf{V} = \dot{w}(t)\mathbf{n}(s),\ \boldsymbol{\omega} = \mathbf{0},$$

where a is the radius of undeformed ring. Let us suppose that

$$\boldsymbol{\mathcal{F}} = \mathbf{N}_d = \mathbf{0}, \quad \boldsymbol{\mathcal{L}} = \mathbf{M}_d = \mathbf{0}, \quad \mathcal{Q} = 0, \quad \vartheta = \mathrm{const}, \quad \eta = \mathrm{const}.$$

The principal axes of inertia of the cross-section of the ring do not coincide with the vectors $\mathbf{n}$ and $\mathbf{b}$

$$\mathbf{d}_1 = \cos\alpha\,\mathbf{n} + \sin\alpha\,\mathbf{b}, \quad \mathbf{d}_2 = -\sin\alpha\,\mathbf{n} + \cos\alpha\,\mathbf{b}. \tag{24}$$

Let us calculate the inertial terms in (11) and (12)

$$\rho_0\dot{\mathbf{V}} = \tilde{\rho}F\ddot{w}\mathbf{n} = -\tilde{\rho}Fa\ddot{w}\,\mathbf{t}', \quad \rho_0\dot{\mathbf{V}}\cdot\boldsymbol{\Theta}_1 = -\ddot{w}\,\mathbf{n}\times\mathbf{d} = -\lambda\ddot{w}\,\mathbf{t} = -a\lambda\ddot{w}\,\mathbf{n}',$$

where

$$\lambda = \tilde{\rho}\frac{\sin 2\alpha}{2a}\int_{(F)}(x^2-y^2)dxdy.$$

The equations of motion (11) and (12) takes a form

$$[\mathbf{N}(s,t) + \tilde{\rho}Fa\ddot{w}(t)\mathbf{t}(s)]' = \mathbf{0},$$

$$\left[\mathbf{M} - \frac{\tilde{\rho}F}{24}(H^2-h^2)\sin 2\alpha\ddot{w}(t)\mathbf{n}(s)\right]' + \left(1 + \frac{\ddot{w}(t)}{a}\right)\mathbf{t}\times\mathbf{N} = \mathbf{0}.$$

From this it follows

$$\mathbf{N}(s,t) = -\tilde{\rho}Fa\ddot{w}(t)\mathbf{t}(s), \quad \mathbf{M} = \frac{\tilde{\rho}F}{24}(H^2-h^2)\sin 2\alpha\ddot{w}(t)\mathbf{n}(s). \tag{25}$$

The first equation in (25) gives the equation of nonlinear oscillator

$$\ddot{w}(t) + f(w) = 0,$$

where the function $f(w)$ is determined by the intrinsic energy. From the eq. (25) the universal constraint

$$24a\,\mathbf{M}\cdot\mathbf{n} + (H^2-h^2)\sin 2\alpha\,\mathbf{N}\cdot\mathbf{t} = 0, \tag{26}$$

follows. This constraint must be valid for any definition of the intrinsic energy. The existing versions of the rod theory do not satisfy constraint (26).

Paradox. It is obvious that the tensor of mirror reflection $\mathbf{Q} = \mathbf{E} - 2\,\mathbf{t}\otimes\mathbf{t}$ must belong to the symmetry grope for all quantities in this problem. However for the vector $\mathbf{N}$ we have $\mathbf{Q}\cdot\mathbf{N} = -\mathbf{N} \neq \mathbf{N}$, i.e. $\mathbf{Q}$ does not belong to the symmetry grope of $\mathbf{N}$. The classical theory of symmetry does not work!

5 The specification of the internal energy

In what follow we shall consider the isothermal processes. The intrinsic energy may be defined as quadratic form

$$\begin{aligned}\rho_0\mathcal{U}(\boldsymbol{\mathcal{E}}_\times, \boldsymbol{\Phi}_\times) &= \mathcal{U}_0 + \mathbf{N}_0\cdot\boldsymbol{\mathcal{E}}_\times + \mathbf{M}_0\cdot\boldsymbol{\Phi}_\times + \\ &+ \frac{1}{2}\underline{\underline{\boldsymbol{\mathcal{E}}_\times\cdot\mathbf{A}\cdot\boldsymbol{\mathcal{E}}_\times}} + \boldsymbol{\mathcal{E}}_\times\cdot\mathbf{B}\cdot\boldsymbol{\Phi}_\times + \frac{1}{2}\underline{\boldsymbol{\Phi}_\times\cdot\mathbf{C}\cdot\boldsymbol{\Phi}_\times} + \boldsymbol{\Phi}_\times\cdot(\boldsymbol{\mathcal{E}}_\times\cdot\mathbf{D})\cdot\boldsymbol{\Phi}_\times,\end{aligned} \tag{27}$$

where the vectors $\mathbf{N}_0$, $\mathbf{M}_0$, second rank tensors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and third rank tensor $\mathbf{D}$ are defined in the reference configuration and are called the elasticity tensors.

The main problem is to find the elasticity tensors. If we take into account only underlined term in (27), then we obtain the classical theory of rods. In some modern versions of the rod theory the twice underlined term is taken into account. All other terms are absent in the existing theories. However, as it will be shown below, no one term in the representation (27) can not be omitted without contradictions. The representation (27) may be rewritten in terms of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$

$$\rho_0 \mathcal{U}(\boldsymbol{\mathcal{E}}_\times, \boldsymbol{\Phi}_\times) = \mathcal{U}_0 + \tilde{\mathbf{N}}_0 \cdot \boldsymbol{\mathcal{E}} + \tilde{\mathbf{M}}_0 \cdot \boldsymbol{\Phi} + \\ + \frac{1}{2} \boldsymbol{\mathcal{E}} \cdot \tilde{\mathbf{A}} \cdot \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}} \cdot \tilde{\mathbf{B}} \cdot \boldsymbol{\Phi} + \frac{1}{2} \boldsymbol{\Phi} \cdot \tilde{\mathbf{C}} \cdot \boldsymbol{\Phi} + \boldsymbol{\Phi} \cdot (\boldsymbol{\mathcal{E}} \cdot \tilde{\mathbf{D}}) \cdot \boldsymbol{\Phi}, \quad (28)$$

where

$$(\tilde{\mathbf{N}}_0, \tilde{\mathbf{M}}_0) = \mathbf{P} \cdot (\mathbf{N}_0, \mathbf{M}_0), \quad (\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}) = \mathbf{P} \cdot (\mathbf{A}, \mathbf{B}, \mathbf{C}) \cdot \mathbf{P}^T, \quad \tilde{\mathbf{D}} = \overset{3}{\underset{1}{\otimes}} \mathbf{P} \odot \mathbf{D}$$

are defined in the actual configuration. Here and in what follows the notation

$$\overset{k}{\underset{1}{\otimes}} \mathbf{P} \odot \mathbf{S} \equiv \overset{k}{\underset{1}{\otimes}} \mathbf{P} \odot \left(S^{i_1 \ldots i_k} \mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_k} \right) \equiv S^{i_1 \ldots i_k} \mathbf{P} \cdot \mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{P} \cdot \mathbf{e}_{i_k}$$

is used for the tensor $\mathbf{S}$ of the rank k.

Let us consider the generalized theory of the tensor symmetry [1]. In our case oriented 3d-space $E_3^{(o)}$ is the direct sum of oriented 2d-space $E_2^{(o)}$ and oriented 1d-space $E_1^{(o)}$

$$E_3^{(o)} = E_1^{(o)} \oplus E_2^{(o)}.$$

Orientations in $E_3^{(o)}$ and $E_1^{(o)}$ may be chosen independently.

Definition: *objects that do not depend on the choice of orientation in* $E_3^{(o)}$ *and* $E_1^{(o)}$ *are called polar ones; objects that depend on the choice of orientation in* $E_3^{(o)}$ *and do not depend on the choice of orientation in* $E_1^{(o)}$ *are called axial ones; objects that do not depend on the choice of orientation in* $E_3^{(o)}$ *but depend on the choice of orientation in* $E_1^{(o)}$ *are called polar* t*-oriented ones; objects that depend on the choice of orientation both in* $E_3^{(o)}$ *and in* $E_1^{(o)}$ *are called axial* t*-oriented ones.*

In theory under consideration: ρ_0, ϑ, η, $\mathcal{U}$, $\mathbf{r}$, $\mathbf{R}$, $\mathbf{u}$, $\boldsymbol{\mathcal{F}}$, $\mathbf{a}_c$, $\mathbf{d}$, $\mathbf{P}$, $\boldsymbol{\Theta}_2$, $\mathbf{A}$, $\mathbf{C}$ are polar objects; R_t, $\boldsymbol{\psi}$, $\boldsymbol{\omega}$, $\boldsymbol{\mathcal{L}}$, $\boldsymbol{\Theta}_1$, $\mathbf{B}$ are axial objects; R_c, $\mathbf{N}_0, \mathbf{N}$, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{E}}_\times$, $\mathbf{D}$ are polar t-oriented objects; $\mathbf{q}$, $\boldsymbol{\tau}$, $\mathbf{M}_0, \mathbf{M}$, $\boldsymbol{\Phi}$, $\boldsymbol{\Phi}_\times$ are axial t-oriented objects. Let us note that the differentiation with respect to s changes the type of an object. For example, $\mathbf{N}$ is the polar t-oriented vector but $\mathbf{N}'$ is the polar vector.

Definition: *the* k*-rank tensor* $\mathbf{S}'$ *is called orthogonal transformation of the* k*-rank tensor* $\mathbf{S}$ *and is defined as*

$$\mathbf{S}' \equiv (\mathbf{t} \cdot \mathbf{Q} \cdot \mathbf{t})^\beta (\det \mathbf{Q})^\alpha \overset{k}{\underset{1}{\otimes}} \mathbf{Q} \odot \mathbf{S}, \quad (29)$$

where $\alpha = 0$, $\beta = 0$, *if* $\mathbf{S}$ *is a polar tensor;* $\alpha = 1$, $\beta = 0$, *if* $\mathbf{S}$ *is an axial tensor;* $\alpha = 0$, $\beta = 1$, *if* $\mathbf{S}$ *is a polar* t*-oriented tensor;* $\alpha = 1$, $\beta = 1$, *if* $\mathbf{S}$ *is an axial* t*-oriented tensor.*

Definition*: the set of the orthogonal solutions of the equation*

$$\mathbf{S}' = \mathbf{S}, \tag{30}$$

is called the symmetry grope of the tensor $\mathbf{S}$*, where* $\mathbf{S}$ *is given and orthogonal tensors* $\mathbf{Q}$ *must be found. The* $\mathbf{S}'$ *is defined by (29).*

Now we are able to explain paradox of the previous section. Vector $\mathbf{N}$ is a polar t-oriented vector. Therefore its symmetry grope must be found from the equation

$$(\mathbf{t} \cdot \mathbf{Q} \cdot \mathbf{t})\, \mathbf{Q} \cdot \mathbf{N} = \mathbf{N}.$$

It is easy to see that the tensor of mirror reflection $\mathbf{Q} = \mathbf{E} - 2\,\mathbf{t} \otimes \mathbf{t}$ belongs to the symmetry grope of $\mathbf{N}$ accordingly to the definition (30).

The requirements of symmetry are necessary tools. However they are not sufficient in order to construct the elasticity tensors. The latter depend on many factors. Even in the simplest case, when the rod made of isotropic material, the elasticity tensors depend on the shape of rod, i.e. on vectors Darboux $\boldsymbol{\tau}$ and $\mathbf{q}$ or, what is the same, on vector $\boldsymbol{\tau}$ and on the intensity of angle of natural twisting φ'. If the diameter of the cross-section is chosen as an unit of length, then the modulus of the vector $\boldsymbol{\tau}$ is a small quantity. By this reason it is possible to use the decomposition

$$\mathbf{f} = \mathbf{f}_0 + \mathbf{f}_1 \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{f}_2 \cdot \boldsymbol{\tau},$$

where $\mathbf{f}$ is any tensor of elasticity.

Making use of this technics one may obtain

$$\mathbf{A} = A_1 \mathbf{d}_1\, \mathbf{d}_1 + A_2 \mathbf{d}_2\, \mathbf{d}_2 + A_3 \mathbf{d}_3\, \mathbf{d}_3 + \frac{A_{12}}{R_t} (\mathbf{d}_1\, \mathbf{d}_2 + \mathbf{d}_2\, \mathbf{d}_1) + \\ + \frac{1}{R_c} \left[A_{13} (\mathbf{d}_1\, \mathbf{d}_3 + \mathbf{d}_3\, \mathbf{d}_1) \cos\alpha + A_{23} (\mathbf{d}_2\, \mathbf{d}_3 + \mathbf{d}_3\, \mathbf{d}_2) \sin\alpha\right], \tag{31}$$

where the meaning of the angle α is defined by (24), $\mathbf{d}_3 \equiv \mathbf{t}$, $\mathbf{a}\,\mathbf{b} \equiv \mathbf{a} \otimes \mathbf{b}$. The representation for $\mathbf{C}$

$$\mathbf{C} = C_1 \mathbf{d}_1\, \mathbf{d}_1 + C_2 \mathbf{d}_2\, \mathbf{d}_2 + C_3 \mathbf{d}_3\, \mathbf{d}_3 + \frac{C_{12}}{R_t} (\mathbf{d}_1\, \mathbf{d}_2 + \mathbf{d}_2\, \mathbf{d}_1) + \\ + \frac{1}{R_c} \left[C_{13} (\mathbf{d}_1\, \mathbf{d}_3 + \mathbf{d}_3\, \mathbf{d}_1) \cos\alpha + C_{23} (\mathbf{d}_2\, \mathbf{d}_3 + \mathbf{d}_3\, \mathbf{d}_2) \sin\alpha\right]. \tag{32}$$

If the natural twisting is absent, then

$$A_{12} = A_{13} = A_{23} = 0, \quad C_{12} = C_{13} = C_{23} = 0.$$

A general representation for $\mathbf{B}$

$$\mathbf{B} = \varphi' B_0 \mathbf{t}\,\mathbf{t} + \frac{1}{R_t}\left[B_1\mathbf{d}_1\mathbf{d}_1 + B_2\mathbf{d}_2\mathbf{d}_2 + B_3\mathbf{t}\,\mathbf{t} + \varphi'(b_1\mathbf{d}_1\mathbf{d}_1 + b_2\mathbf{d}_2\mathbf{d}_2)\times\mathbf{t}\right] +$$
$$+\frac{1}{R_c}\left[(B_{13}\mathbf{d}_1\,\sin\alpha + B_{23}\mathbf{d}_2\,\cos\alpha)\,\mathbf{t} + \mathbf{t}\,(B_{31}\,\mathbf{d}_1\sin\alpha + B_{32}\mathbf{d}_2\cos\alpha)\right] +$$
$$+\frac{\varphi'}{R_c}\left[(b_{13}\mathbf{d}_1\cos\alpha + b_{23}\mathbf{d}_2\sin\alpha)\,\mathbf{t} + \mathbf{t}\,(b_{31}\mathbf{d}_1\cos\alpha + b_{32}\mathbf{d}_2\sin\alpha)\right]. \quad (33)$$

If the natural twisting is absent, then $\varphi' = 0$. Not all elastic modulus in (33) are important. In order to see that fact let us write down the expression

$$\boldsymbol{\mathcal{E}}\cdot\mathbf{B} = \varphi' B_0\,\epsilon\mathbf{t} + \frac{1}{R_t}\left[B_1\Gamma_1\mathbf{d}_1 + B_2\Gamma_2\mathbf{d}_2 + B_3\epsilon\,\mathbf{t} + \varphi'(b_1\Gamma_1\mathbf{d}_1 + b_2\Gamma_2\mathbf{d}_2)\times\mathbf{t}\right] +$$
$$+\frac{1}{R_c}\left[(B_{13}\Gamma_1\,\sin\alpha + B_{23}\Gamma_2\,\cos\alpha)\,\mathbf{t} + \epsilon\,(B_{31}\,\mathbf{d}_1\sin\alpha + B_{32}\mathbf{d}_2\cos\alpha)\right] +$$
$$+\frac{\varphi'}{R_c}\left[(b_{13}\Gamma_1\cos\alpha + b_{23}\Gamma_2\sin\alpha)\,\mathbf{t} + \epsilon\,(b_{31}\mathbf{d}_1\cos\alpha + b_{32}\mathbf{d}_2\sin\alpha)\right]. \quad (34)$$

Because the shear deformations Γ_1, Γ_2 are as a rule small then instead of (34) one may write

$$\boldsymbol{\mathcal{E}}\cdot\mathbf{B} = \varphi' B_0\,\epsilon\mathbf{t} + \frac{\epsilon}{R_t}B_3\,\mathbf{t} +$$
$$+\frac{\epsilon}{R_c}(B_{31}\,\mathbf{d}_1\sin\alpha + B_{32}\mathbf{d}_2\cos\alpha) + \frac{\epsilon\varphi'}{R_c}(b_{31}\mathbf{d}_1\cos\alpha + b_{32}\mathbf{d}_2\sin\alpha). \quad (35)$$

Thus we see that only modulus B_0, B_3, B_{31}, B_{32}, b_{31}, b_{32} may be important. More over it is clear from physical sense that modulus b_{31}, b_{32} may be ignored as well. Thus instead of (33) one may write down

$$\mathbf{B} = \varphi' B_0\,\mathbf{t}\,\mathbf{t} + \frac{B_3}{R_t}\mathbf{t}\,\mathbf{t} + \frac{1}{R_c}\mathbf{t}\,(B_{31}\,\mathbf{d}_1\sin\alpha + B_{32}\mathbf{d}_2\cos\alpha).$$

This technology does not suit in order to find the vectors $\mathbf{N}_0$ and $\mathbf{M}_0$, which linearly depend on the external loads. As a rule these vectors are not important.

6 The determination of the elastic modulus

At the present time all elastic modulus have been found. Let us show how to find the elastic modulus A_1, A_2, A_3. It is easy to prove the representations

$$A_3 = E\,F, \quad A_1 = k_1\,G\,F, \quad A_2 = k_2\,G\,F, \quad (36)$$

where E is the Yang modulus, $G = E/2(1+\nu)$ is the shear modulus of the material of rod.

Dimensionless coefficients k_1 and k_2 in (36) are called the shear correction factors. There are many different values for these factors, but all of them must satisfy the inequality

$$\pi^2/12 \le k_1,\, k_2 < 1.$$

In order to illustrate the determination of shear correction factor let us consider the next dynamics problem of 3d-theory of elasticity for the body occupying the domain: $-h/2 \le x \le h/2$, $-H/2 \le y \le H/2$, $0 \le z \le l$. Let us accept that $\mathbf{i} = \mathbf{d}_1$, $\mathbf{j} = \mathbf{d}_2$, $\mathbf{k} = \mathbf{t}$. Let the lateral surface of the body is free. The boundary conditions are determined as

$$z = 0,\, l: \quad \mathbf{u}_{(3)} \cdot \mathbf{d}_1 = \mathbf{u}_{(3)} \cdot \mathbf{d}_2 = 0, \quad \mathbf{t} \cdot \mathbf{T} \cdot \mathbf{t} = 0,$$

where $\mathbf{u}_{(3)}$ and $\mathbf{T}$ are the vector of displacement and the stress tensor respectively. Let us consider the shear vibrations of the form

$$\mathbf{u}_{(3)} = W e^{i\omega t} \sin \lambda x\, \mathbf{t}, \quad \mathbf{T} = G\lambda W e^{i\omega t} \cos \lambda x (\mathbf{t}\, \mathbf{d}_1 + \mathbf{d}_1\, \mathbf{t}), \quad \lambda = (2k+1)\pi/h,$$

where ω is the natural frequency of the body. These expressions satisfy the boundary conditions. To satisfy the equations of motion we have to accept

$$\boldsymbol{\nabla} \cdot \mathbf{T} = \tilde{\rho} \ddot{\mathbf{u}}_{(3)} \quad \Rightarrow \quad \omega^2 = \frac{G}{\tilde{\rho}} \frac{(2k+1)^2 \pi^2}{h^2}, \quad k = 0, 1, 2, \ldots \tag{37}$$

Let us consider this in the framework of the beam theory. We have

$$\mathbf{u} = \mathbf{0}, \quad \boldsymbol{\psi} = \psi_2 \mathbf{d}_2 = const, \quad \mathbf{N} = N_1 \mathbf{d}_1, \quad \mathbf{M} = \mathbf{0}, \quad \mathbf{N}_0 = \mathbf{0}, \quad \mathbf{M}_0 = \mathbf{0};$$

$$\mathbf{e} \equiv \mathbf{u}' + \mathbf{t} \times \boldsymbol{\psi} = -\psi_2 \mathbf{d}_1, \quad \boldsymbol{\kappa} \equiv \boldsymbol{\psi}' = \mathbf{0}, \quad \mathbf{N} = -A_1 \psi_2 \mathbf{d}_1, \quad \mathbf{M} = \mathbf{0}.$$

The equation of motion takes a form

$$\mathbf{N}'(s,t) = \mathbf{0}, \quad -A_1 \psi_2 \mathbf{d}_2 = \Theta_2 \ddot{\psi}_2 \mathbf{d}_2 \;\Rightarrow\; \omega^2 = A_1/\Theta_2, \quad \Theta_2 = \tilde{\rho} F h^2/12. \tag{38}$$

Comparing the frequencies found in terms of the three-dimensional theory (37), and the frequency found under the theory of beam (38), we see huge distinction. The three-dimensional theory gives the spectrum of the natural frequencies while the beam theory gives only one frequency. It is not surprising, for area of applicability of the three-dimensional theory is much more wider than area of applicability of the beam theory. The beam theory gives a good description only low-frequency vibrations. Let us note, that shift vibrations are already high-frequency vibrations, their frequencies trend to infinity at $h \to 0$. While frequencies of bending vibrations trend to zero at $h \to 0$, and frequencies of longitudinal vibrations are limited at $h \to 0$. Therefore it is quite natural, that the beam theory does not allow to describe all shift spectrum, but it can describe the lowest frequency from a spectrum (37). For this end it is enough to accept

$$\frac{A_1}{\Theta_2} = \frac{G}{\tilde{\rho}} \frac{\pi^2}{h^2} \quad \Rightarrow \quad A_1 = \frac{\pi^2}{12} GF \quad \Rightarrow \quad k_1 = \frac{\pi^2}{12}.$$

It may be proved that $k_1 = k_2$.

It is useful to consider the certain seeming paradox connected to definition of shear coefficient. We shall try to determine it from the exact decision of a static problem on pure shift of a beam, which is given by formulas

$$\mathbf{T} = \tau\,(\mathbf{t}\,\mathbf{d}_1 + \mathbf{d}_1\,\mathbf{t}), \quad G\mathbf{u}_{(3)} = \tau\,x\,\mathbf{t} \quad \Rightarrow$$

$$\Rightarrow \quad \mathbf{N} = \tau\,F\,\mathbf{d}_1, \quad \mathbf{M} = \mathbf{0}, \quad \mathbf{u} = \mathbf{0}, \quad G\,\boldsymbol{\psi} = -\tau\,\mathbf{d}_2.$$

From the other hand we have

$$\mathbf{N} = \mathbf{A}\cdot(\mathbf{t}\times\boldsymbol{\psi}) \quad \Rightarrow \quad \tau\,F = -A_1\,\mathbf{d}_2\cdot\boldsymbol{\psi} \quad \Rightarrow \quad k_1 = 1. \tag{39}$$

Namely this value of shear coefficient was obtained by M. Rubin (2003). His arguments are based on the solution (39). Thus we obtain a theoretical paradox: from two exact solution we obtain two different values of shear coefficient. The existing beam theory are not able to explain this paradox.

In fact the solution of this paradox is very simple. Let us consider the expression (27). It contains the vectors $\mathbf{N}_0$ and $\mathbf{M}_0$, which are linear functions of loads acting on lateral surface of beam. Because of this the equality (39) must be written as

$$\mathbf{N} = \mathbf{N}_0 + \mathbf{A}\cdot(\mathbf{t}\times\boldsymbol{\psi}) \quad \Rightarrow \quad \mathbf{N}_0 = \tau\,F\,(1-k_1)\mathbf{d}_1.$$

Therefore the problem on pure shear does not allow to calculate the shear coefficient.

The elastic modulus C_3, C_1, C_2 are well known

$$C_1 = E\,J_1, \quad C_2 = E\,J_2, \quad J_1 \equiv \int\limits_{(F)} y^2 dxdy, \quad J_2 \equiv \int\limits_{(F)} x^2 dxdy, \tag{40}$$

$$C_3 = G\,J_r, \quad J_r = 2\int\limits_{(F)} U(x,y)dxdy, \quad \triangle U = -2, \quad U = 0 \text{ on } \partial F. \tag{41}$$

Let us consider the tensor of elasticity $\mathbf{B}$. In known versions of the rod theory we have

$$B_{31} = 0, \quad B_{32} = 0, \quad B_3 = 0.$$

However the representation (34) and universal equality (26) it follows

$$B_{32} = EJ_4 + B_{31}, \quad J_4 \equiv \int (x^2 - y^2)dxdy \neq 0. \tag{42}$$

Thus the conditions $B_{32} = B_{31} = 0$ are impossible. The next formulas may be proved

$$B_0 = E\,(J_1 + J_2 - J_r) \geq 0, \quad B_{32} = C_2, \quad B_{31} = C_1. \tag{43}$$

The above presented rod theory is consistent nonlinear theory with very wide branch of applicability. At present author does not know the problems when this theory leads to some contradictions or mistakes.

7 The longitudinal-twisting waves in the rod

Let us consider the longitudinal-twisting waves in the naturally twisted beam.

$$\frac{\partial^2 u}{\partial s^2} - \frac{1}{c_l^2}\frac{\partial^2 u}{\partial t^2} + \frac{\varphi' B_0}{EF}\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{c_l^2}\mathcal{F}_t = 0, \qquad c_l^2 = \frac{E}{\rho}. \tag{44}$$

$$\frac{\partial^2 \psi}{\partial s^2} - \frac{1}{c_t^2}\frac{\partial^2 \psi}{\partial t^2} + \frac{\varphi' B_0}{GJ_r}\frac{\partial^2 u}{\partial s^2} + \frac{\rho F}{GJ_r}\mathcal{L}_t = 0, \quad c_t^2 = \frac{GJ_r}{\rho J_p}. \tag{45}$$

The solution of the system (44)–(45) may be represented in terms of solutions of the wave equations

$$\frac{\partial^2 \nu}{\partial s^2} - \frac{1}{\Omega_1}\frac{\partial^2 \nu}{\partial t^2} = 0, \quad \frac{\partial^2 \vartheta}{\partial s^2} - \frac{1}{\Omega_2}\frac{\partial^2 \vartheta}{\partial t^2} = 0, \tag{46}$$

where Ω_1 and Ω_2 are some parameters, which must be found. A general solution of the system (44)–(45) has a form

$$u(s,t) = \nu(s,t) + \frac{\gamma_1 c_l^2}{\Omega_2 - c_l^2}\vartheta(s,t), \quad \psi(s,t) = \vartheta(s,t) + \frac{\gamma_2 c_t^2}{\Omega_1 - c_t^2}\nu(s,t), \tag{47}$$

where

$$\gamma_1 \equiv \frac{\varphi' B_0}{EF}, \quad \gamma_2 \equiv \frac{\varphi' B_0}{GJ_r},$$

ν and ϑ are solutions of (46). The parameters Ω_1 and Ω_2 are the roots of equation

$$\Omega_1^2 - (c_l^2 + c_t^2)\Omega_1 + (1 - \gamma_1\gamma_2)c_l^2 c_t^2 = 0, \quad \Omega_2 < c_t^2, \quad \Omega_1 > c_l^2, \quad c_t^2 < c_l^2.$$

So, the presence of natural twisting in a beam does not change a character of wave process in the beam. It still waves without a dispersion, but the presence of natural twisting changes velocities of wave propagation in a beam. The longitudinal - torsional wave is the solution of the first equation from (46), and the velocity of its propagation $\sqrt{\Omega_1}$ is bigger than the velocity of propagation of longitudinal wave in a beam without natural twisting. The torsional-longitudinal wave is the solution of second equation from (46), and the velocity of its propagation $\sqrt{\Omega_2}$ appears below velocity of propagation of a wave of torsion in a beam without natural twisting.

8 The twisting of a beam by the dead moments

In order to show how to work with presented rod theory let us consider the task of twisting of beam by the dead moment when external surface loads in (11)–(12) are absent, i.e. $\mathbf{F} = \mathbf{0}$, $\mathbf{L} = \mathbf{0}$. The equation of equilibrium

$$\mathbf{N}'(s,t) = \mathbf{0}, \quad \mathbf{M}' + \mathbf{R}' \times \mathbf{N} = \mathbf{0}. \tag{48}$$

The boundary conditions

$$s = 0: \ \mathbf{R} = \mathbf{0}, \quad \mathbf{P} = \mathbf{E}; \qquad s = l: \ \mathbf{N} = \mathbf{0}, \quad \mathbf{M} = \mathbf{L} \equiv L\,\mathbf{m}, \tag{49}$$

where $\mathbf{L} = \mathsf{const}$ and $\mathbf{m}$ is unit constant vector. Solution of static equations (48) taking into account boundary conditions (49)

$$\mathbf{N} = \mathbf{0}, \quad \mathbf{M} = \mathbf{L} = L\,\mathbf{m}. \tag{50}$$

Cauchy-Green relations of naturally twisted beam

$$\mathbf{N} = \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{P}^T \cdot \mathcal{E} + \varphi' B_0 (\mathbf{t} \cdot \mathbf{P}^T \cdot \mathbf{\Phi}) \mathbf{P} \cdot \mathbf{t},$$

$$\mathbf{M} = \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^T \cdot \mathbf{\Phi} + \varphi' B_0 (\mathcal{E} \cdot \mathbf{P} \cdot \mathbf{t}) \mathbf{P} \cdot \mathbf{t},$$

where φ' is the natural twisting of beam: $\varphi = 2\pi s/a$, a is a length on which the cross-section is turning by the angle 2π. We see that

$$\mathcal{E} = -\left(\frac{\varphi' B_0}{A_3}\right)(\mathbf{\Phi} \cdot \mathbf{P} \cdot \mathbf{t}) \mathbf{P} \cdot \mathbf{t} \quad \Rightarrow \quad \mathbf{R}' = \left(1 - \frac{\varphi' B_0}{A_3} \mathbf{\Phi} \cdot \mathbf{P} \cdot \mathbf{t}\right) \mathbf{P} \cdot \mathbf{t}, \tag{51}$$

$$\mathbf{L} = \mathbf{P} \cdot [C_t \mathbf{t}\,\mathbf{t} + C_1 \mathbf{d}_1\,\mathbf{d}_1 + C_2 \mathbf{d}_2\,\mathbf{d}_2] \cdot \mathbf{P}^T \cdot \mathbf{\Phi}, \quad C_t \equiv C_3 \left(1 - \frac{\varphi'^2 B_0^2}{C_3 A_3}\right). \tag{52}$$

Let us accept that $C_1 = C_2$. Then from (52) it follows

$$\mathbf{\Phi} = \mathbf{P} \cdot [C_t^{-1} \mathbf{t}\,\mathbf{t} + C_1^{-1} (\mathbf{E} - \mathbf{t}\,\mathbf{t})] \cdot \mathbf{P}^T \cdot \mathbf{L}, \quad \mathbf{P}' = \mathbf{\Phi} \times \mathbf{P}. \tag{53}$$

The system (53) has a first integral

$$\mathbf{\Phi} \cdot \mathbf{L} = \mathbf{L} \cdot \mathbf{P} \cdot [C_t^{-1} \mathbf{t}\,\mathbf{t} + C_1^{-1} (\mathbf{E} - \mathbf{t}\,\mathbf{t})] \cdot \mathbf{P}^T \cdot \mathbf{L} = \mathsf{const}. \tag{54}$$

The energy integral (54) is the constraint on the turn-tensor $\mathbf{P}$. A general solution of (54) has a form

$$\mathbf{P}(t) = \mathbf{Q}(\alpha\,\mathbf{m}) \cdot \mathbf{Q}(\beta\,\mathbf{t}), \tag{55}$$

where notation

$$\mathbf{Q}(\gamma\,\mathbf{p}) \equiv (1 - \cos\gamma) \mathbf{p}\ \mathbf{p} + \cos\gamma\,\mathbf{E} + \sin\gamma\ \mathbf{p} \times \mathbf{E}$$

is used for rotation by the angle γ around unit vector $\mathbf{p}$. For any $\alpha(s)$ and $\beta(s)$ the energy (54) keeps a constant value. Making use of (55), the system (53) rewrite in a form

$$\mathbf{\Phi} = \mathbf{Q}(\alpha\,\mathbf{m}) \cdot [C_t^{-1} \mathbf{t}\,\mathbf{t} + C_1^{-1} (\mathbf{E} - \mathbf{t}\,\mathbf{t})] \cdot \mathbf{L}, \quad \mathbf{\Phi} = \mathbf{Q}(\alpha\,\mathbf{m}) \cdot (\alpha' \mathbf{m} + \beta' \mathbf{t})$$

or

$$\alpha'(s)\,\mathbf{m} + \beta'(s)\,\mathbf{t} = L\,[C_t^{-1} \mathbf{t}\,\mathbf{t} + C_1^{-1} (\mathbf{E} - \mathbf{t}\,\mathbf{t})] \cdot \mathbf{m} = L\,[(C_t^{-1} - C_1^{-1}) \cos\sigma\ \mathbf{t} + C_1^{-1}\,\mathbf{m}].$$

The solution of this system

$$\alpha'(s) = L\,C_1^{-1}, \quad \beta'(s) = L\,(C_t^{-1} - C_1^{-1}) \cos\sigma, \quad \cos\sigma \equiv \mathbf{m} \cdot \mathbf{t}. \tag{56}$$

From (56) it follows

$$\alpha(s) = L\,C_1^{-1} s, \quad \beta(s) = L \cos\sigma\,(C_t^{-1} - C_1^{-1})\,s.$$

It is easy to calculate

$$\mathbf{\Phi}\cdot\mathbf{P}\cdot\mathbf{t} = \frac{L\cos\sigma}{C_t}. \tag{57}$$

This is a variation of twisting of the beam

$$\tilde{\mathbf{q}}\cdot\mathbf{P}\cdot\mathbf{t} = \varphi' + \mathbf{\Phi}\cdot\mathbf{P}\cdot\mathbf{t}.$$

The axis extension follows from (51)

$$\varepsilon = \boldsymbol{\mathcal{E}}\cdot\mathbf{P}\cdot\mathbf{t} = -\frac{\varphi' B_0}{A_3}\,\frac{L\cos\sigma}{C_t}. \tag{58}$$

If $\varphi' L > 0$, then $\varepsilon < 0$. If $\varphi' L < 0$, then $\varepsilon > 0$.

Let us calculate the Darboux vector, curvature and twisting of deformed beam

$$\tilde{\tau} = \alpha'\cos\sigma\,\tilde{\mathbf{t}} - \alpha'\sin\sigma\,\tilde{\mathbf{b}} = \alpha'\mathbf{m} \quad\Rightarrow\quad \tilde{R}_c^{-1} = \alpha'\sin\sigma, \quad \tilde{R}_t^{-1} = \alpha'\cos\sigma.$$

In order to find the actual configuration of the beam it is necessary to integrate (51)

$$\mathbf{R} = (1+\varepsilon)\left[s\cos\sigma\,\mathbf{m} + \frac{C_1}{L}\mathbf{Q}\left(\frac{Ls}{C_1}\mathbf{m}\right)\cdot(\mathbf{t}\times\mathbf{m}) - \frac{C_1}{L}(\mathbf{t}\times\mathbf{m})\right].$$

The vector in square brackets of this expression, describes a spiral on the cylinder of radius $R_0 = C_1\sin\sigma/|L|$. The axis of the cylinder is spanned on a vector $\mathbf{m}$ and passes through the point determined by a vector

$$(1+\varepsilon)C_1(\pi\cos\sigma\,\mathbf{m} - 2\mathbf{t}\times\mathbf{m})/L.$$

The length of one coil of a spiral is equal $2\pi C_1/|L|$. The step h of a spiral is equal $l\cos\sigma$.

9 Elastica of Euler (1744)

Mathematical statement

$$\mathbf{N}' = \mathbf{0}, \quad \mathbf{M}' + \mathbf{R}'\times\mathbf{N} = \mathbf{0}; \tag{59}$$

$$\mathbf{M} = \mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}\cdot\mathbf{\Phi} = (C_3 - C_1)(\mathbf{t}\cdot\mathbf{P}^T\cdot\mathbf{\Phi})\mathbf{P}\cdot\mathbf{t} + C_1\mathbf{\Phi}, \tag{60}$$

where $\mathbf{N}$ is defined by equation of equilibrium. Boundary conditions

$$s = 0:\ \mathbf{R} = \mathbf{0}, \quad \mathbf{P} = \mathbf{E}; \qquad s = l:\ \mathbf{N} = -N\mathbf{t}, \quad \mathbf{M} = \mathbf{0}. \tag{61}$$

Kinematic relations

$$\mathbf{R}' = \mathbf{P}\cdot\mathbf{t}, \quad \mathbf{P}' = \mathbf{\Phi}\times\mathbf{P}, \quad |\mathbf{R}'| = 1.$$

The problem (59)–(61) has an obvious solution

$$\mathbf{R}(s) = s\,\mathbf{t}, \quad \mathbf{P} = \mathbf{E}, \quad \mathbf{N} = -N\mathbf{t}, \quad \mathbf{M} = \mathbf{0}. \tag{62}$$

As it was shown by Euler the solution (62) is unique solution if $N \le N_{cr}$. If $N > N_{cr}$, then there are another solutions. It is possible to prove that all this solutions are plain curves. Beside for the vector the next representation

$$\mathbf{\Phi} = \mathbf{R}'\times\mathbf{R}'' = -R_c^{-1}\mathbf{b} \equiv \psi'(s)\mathbf{b}, \quad \mathbf{b} \equiv \tilde{\mathbf{b}}, \quad \psi'(s) \equiv -R_c^{-1}(s) \tag{63}$$

may be found. In such a case the turn-tensor has a form

$$\mathbf{P} = \mathbf{Q}(\psi \mathbf{b}) \quad \Rightarrow \quad \mathbf{R}' = \cos\psi(s)\,\mathbf{t} + \sin\psi(s)\,\mathbf{b}. \tag{64}$$

Thus we have

$$\mathbf{N} = -N\mathbf{t}, \quad \mathbf{M} = \mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}\cdot\mathbf{\Phi} = C_1\,\psi'(s)\mathbf{b}. \tag{65}$$

For determination $\psi(s)$ we have well-known boundary value problem

$$C_1\psi'' + N\sin\psi = 0, \quad \psi(0) = 0, \quad \psi'(l) = 0. \tag{66}$$

If

$$N > N_{cr} \equiv \frac{\pi^2 C_2}{4l^2},$$

then (66) has nontrivial solutions. The exact solution of (66) is well-known. Let us show the approximate solution for small values of $\gamma \equiv 1 - \sqrt{N_{cr}/N} > 0$

$$\psi(s) = \psi_l \sin\vartheta = 4\left(\sqrt{\frac{N}{N_{cr}}} - 1\right)^{1/2} \sin\left[\frac{\pi s}{2l} + \frac{\gamma}{2}\sin\frac{\pi s}{l}\right]. \tag{67}$$

Let's sum up. If longitudinal stretching force is applied to the free end of a beam, then there is only one rectilinear equilibrium configuration. The situation varies, if on a beam acts compressing force. In this case always there is a rectilinear equilibrium configuration, which is determined by the following expressions

$$\mathbf{R}(s) = (1 - N/A)\,s\mathbf{t}, \quad \mathbf{P} = \mathbf{E}, \quad \mathbf{N} = -N\mathbf{t}, \quad \mathbf{M} = \mathbf{0}. \tag{68}$$

If the module of compressing force N exceeds value Euler's critical force N_{cr}, then there is one more solution submitted by the formula (67). Intuitively clearly, that at $N > N_{cr}$ the second solution is realized. The first solution will be unstable.

In the literature [2] at judgement about stability of an equilibrium configuration usually use the energetic reasons. Namely, the stable configuration is supposed to be those that has smaller energy. Strictly speaking, comparison of energies of equilibrium configurations have no the direct relation to concept of stability. An equilibrium configuration of conservative system is steady, if its potential energy has an isolated local minimum, which is not connected to energy of other equilibrium configuration. Nevertheless, from two possible equilibrium configurations the Nature if it is possible, chooses a configuration with smaller energy. Therefore in Euler's elastica it supposed that the bent configuration is stable, as potential energy in this case is less [2]. Nevertheless, a such arguments in Euler's elastica are not valid. The matter is that in a considered case a minimum of energy is not isolated. Actually we have family of the equilibrium bent configurations, and all of them possess the same energy. Really, the received decision allows to find an angle of turn unequivocally ψ around of a vector of a binormal $\mathbf{b}$, but the vector $\mathbf{b}$ has not been determined uniquely manner, for it is possible rotate $\mathbf{b}$ around $\mathbf{t}$

$$\mathbf{b} = \mathbf{Q}\left(\varphi(t)\mathbf{t}\right)\cdot\mathbf{b}_0,$$

where $\mathbf{b}_0$ is an arbitrary fixed vector orthogonal $\mathbf{t}$; $\varphi(t)$ is the arbitrary angle of turn around $\mathbf{t}$. From this it follows

$$\mathbf{P}(s) = \mathbf{Q}(\varphi\mathbf{t}) \cdot \mathbf{Q}(\psi\mathbf{b}_0) \cdot \mathbf{Q}^{\mathsf{T}}(\varphi\mathbf{t}) \quad \Rightarrow \quad \mathbf{P}\big|_{s=0} = \mathbf{E};$$

$$\mathbf{R}' = \mathbf{Q}(\varphi\mathbf{t}) \cdot [\cos\psi(s)\,\mathbf{t} + \sin\psi(s)\,\mathbf{b}_0].$$

If φ depends on time, then an angular velocity may be calculated as [3]

$$\boldsymbol{\omega} = \dot{\varphi}\,[(1 - \cos\psi)\,\mathbf{t} - \sin\psi\mathbf{b} \times \mathbf{t}], \quad \boldsymbol{\omega}\big|_{s=0} = 0.$$

Thus, if in Euler's elastica we give to the bent beam small angular velocity, then it will slowly rotate around of the vector $\mathbf{t}$, running all set of equilibrium configurations. And for this it is not required of application of the external moment. It is necessary to emphasize, that we do not mean the rotations of the beam as the rigid whole. For example, the clamped end of a beam does not turn, for at $s = 0$ the turn-tensor becomes unit tensor for any value of φ. In fact the beam does not resist to special kinds of deformation, that for real beam does not correspond to the reality. Let's note that mentioned fact is present for any form of specific energy of a beam. The only important requirement is that the specific energy must be transversally isotropic. In particular, the marked feature explains the so-called Nikolai paradox [4]. Nikolai shows that the equilibrium configuration of a beam loaded by dead (or following) moment, is unstable for arbitrary small value of moment. This result is in sharp contradiction with experimental data. It is supposed that the Nikolai paradox is due to nonconservativity of problem. However this explanation is unsatisfactory, for it is easy to show, that the Nikolai paradox exists in a problem on twisting of a beam by the potential (conservative) moment.

10 Stationary rotations in the Euler elastica

Below the Euler elastica will be examined in dynamic statement. The equation of motion

$$\mathbf{N}'' = \rho F\ddot{\mathbf{R}}', \quad \mathbf{M}' + \mathbf{R}' \times \mathbf{N} = \mathbf{0}, \quad \mathbf{M} = C_1\,\mathbf{R}' \times \mathbf{R}'', \quad \mathbf{R}' = \mathbf{P} \cdot \mathbf{t}. \tag{69}$$

The boundary conditions (61)

$$s = 0: \ \mathbf{R} = \mathbf{0}, \quad \mathbf{P} = \mathbf{E}, \quad \mathbf{N}' = \mathbf{0}; \qquad s = l: \ \mathbf{N} = -N\mathbf{t}, \quad \mathbf{M} = \mathbf{0}. \tag{70}$$

Let's look for solution of the task (69)–(70) in a form

$$\mathbf{P}(s,t) = \mathbf{Q}\,[\varphi(t)\mathbf{t})] \cdot \mathbf{Q}\,[\psi(s)\mathbf{e})] \cdot \mathbf{Q}^{\mathsf{T}}\,[\varphi(t)\mathbf{t})], \quad \mathbf{e} \cdot \mathbf{t} = 0, \tag{71}$$

where $\mathbf{e}$ is the constant unit vector. The vector of bending-twisting $\boldsymbol{\Phi}$ corresponding to the turn-tensor (71)

$$\boldsymbol{\Phi} = \psi'(s)\mathbf{Q}\,[\varphi(t)\mathbf{t})] \cdot \mathbf{e} = \psi'(s)\mathbf{e}_*, \quad \mathbf{e}_*(t) \equiv \mathbf{Q}\,[\varphi(t)\mathbf{t})] \cdot \mathbf{e}. \tag{72}$$

Besides there are formulas

$$\mathbf{R}' = \cos\psi(s)\,\mathbf{t} + \sin\psi(s)\,\mathbf{e}_*(t) \times \mathbf{t}, \quad \ddot{\mathbf{R}}' = \sin\psi(s)\left(\ddot{\varphi}\mathbf{e}_*(t) - \dot{\varphi}^2\mathbf{e}_*(t) \times \mathbf{t}\right).$$

For vector $\mathbf{N}$ may be used decomposition

$$\mathbf{N} = -N\mathbf{t} + Q_* \mathbf{e}_* + Q\mathbf{e}_* \times \mathbf{t}, \quad Q'_*(0,t) = Q'(0,t) = 0, \quad Q_*(l,t) = Q(l,t) = 0.$$

Substituting these expressions into the first equation from (69) one will get

$$Q'' = -\rho F \dot{\varphi}^2 \sin\psi, \quad Q''_* = \rho F \ddot{\varphi} \sin\psi. \tag{73}$$

The vector of moment is expressed as

$$\mathbf{M} = C_1 \mathbf{R}' \times \mathbf{R}'' = C_1 \psi' \mathbf{e}_*.$$

The second equation from (69) is equivalent to

$$C_1 \psi'' + N\sin\psi + Q\cos\psi - Q_* \sin\psi = 0, \quad Q_* = 0. \tag{74}$$

From this it follows that in the Euler elastica only the stationary rotations are possible

$$\ddot{\varphi} = 0 \quad \Rightarrow \quad \dot{\varphi} \equiv \omega = \text{const}. \tag{75}$$

Thus we obtain the next nonlinear boundary value problem

$$Q'' = -\rho F \omega^2 \sin\psi, \quad C_1 \psi'' + N\sin\psi + Q\cos\psi = 0; \tag{76}$$

$$s = 0: \; Q'(0) = 0, \quad \psi(0) = 0; \qquad s = l: \; Q(l) = 0, \quad \psi'(l) = 0. \tag{77}$$

The problem (76)–(77) is difficult to find the exact solution. However it is easy to find the approximated solution for small value of ω^2. Let's use the decomposition

$$\psi(s) = \psi_{st}(s) + \vartheta(s), \quad |\vartheta(s)| \ll 1,$$

where $\psi_{st}(s)$ is the solution of the static task at $N > N_{cr}$. In such a case instead of (76)–(77) we obtain

$$Q'' = -\rho F \omega^2 \sin\psi_{st}, \quad C_1 \vartheta'' + (N\cos\psi_{st})\,\vartheta = Q\cos\psi_{st}; \tag{78}$$

$$s = 0: \; Q'(0) = 0, \quad \vartheta(0) = 0; \qquad s = l: \; Q(l) = 0, \quad \vartheta'(l) = 0. \tag{79}$$

The problem (78)–(79) has unique solution.

Thus, the account of forces of inertia does not change a conclusion about presence rotating "equilibrium" configurations. That means, that the bent equilibrium configurations in the Euler elastica are unstable, for the turned bent configuration is not any more close to the original configurations.

It is necessary to emphasize, that experiment does not confirm a conclusion about presence rotating "equilibrium" configurations. The rough experiment which has been carried out by the author, has shown, that if bent equilibrium configuration slightly to push, low-frequency vibrations start, but not rotations.

11 The Nikolai paradox

11.1 Potential moment

Let us introduce a concept of potential moment. This concept is necessary for a statement and an analysis of many problems. Nevertheless a general definition of potential moment is absent in the literature.

Definition: *A moment* $\mathbf{M}(t)$ *is called potential, if there exists a scalar function* $U(\boldsymbol{\theta})$ *depending on a turn-vector such that*

$$\mathbf{M} \cdot \boldsymbol{\omega} = -\dot{U}(\boldsymbol{\theta}) = -\frac{dU}{d\boldsymbol{\theta}} \cdot \dot{\boldsymbol{\theta}}. \tag{80}$$

One may obtain the equality

$$\dot{\boldsymbol{\theta}}(t) = \mathbf{Z}(\boldsymbol{\theta}) \cdot \boldsymbol{\omega}(t), \tag{81}$$

where

$$\mathbf{Z}(\boldsymbol{\theta}) = \mathbf{E} - \frac{1}{2}\mathbf{R} + \frac{1-g}{\theta^2}\mathbf{R}^2, \quad g = \frac{\theta \sin\theta}{2(1-\cos\theta)}, \quad \theta = |\boldsymbol{\theta}|. \tag{82}$$

The tensor $\mathbf{Z}(\boldsymbol{\theta})$ will be called the integrating tensor in the following. The equality (80) can be rewritten in the form

$$\left(\mathbf{M} + \frac{dU}{d\boldsymbol{\theta}} \cdot \mathbf{Z}\right) \cdot \boldsymbol{\omega} = 0.$$

From this it follows

$$\mathbf{M} = -\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \frac{dU}{d\boldsymbol{\theta}} + \mathbf{f}(\boldsymbol{\theta}, \boldsymbol{\omega}) \times \boldsymbol{\omega}, \tag{83}$$

where $\mathbf{f}(\boldsymbol{\theta}, \boldsymbol{\omega})$ is an arbitrary function of vectors $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$.

Definition: *A moment* $\mathbf{M}$ *is called positional, if* $\mathbf{M}$ *depends on the turn-vector* $\boldsymbol{\theta}$ *only. For the positional moment* $\mathbf{M}(\boldsymbol{\theta})$ *we have*

$$\mathbf{M}(\boldsymbol{\theta}) = -\mathbf{Z}^T(\boldsymbol{\theta}) \cdot \frac{dU(\boldsymbol{\theta})}{d\boldsymbol{\theta}}. \tag{84}$$

Definition: *The potential* $U(\boldsymbol{\theta})$ *is called transversally isotropic with an axis of symmetry* $\mathbf{k}$*, if the equality*

$$U(\boldsymbol{\theta}) = U[\mathbf{Q}(\alpha\mathbf{k}) \cdot \boldsymbol{\theta}]$$

holds for any turn-tensor $\mathbf{Q}(\alpha\mathbf{k})$.

It can be proved that a general form of a transversally isotropic potential can be expressed as a function of two arguments

$$U(\theta) = F\left(\mathbf{k} \cdot \theta, \theta^2\right). \tag{85}$$

For this potential one can derive the expression

$$\mathbf{M}(\theta) = -2\frac{\partial F}{\partial(\theta^2)}\theta - \frac{\partial F}{\partial(\mathbf{k} \cdot \theta)}\mathbf{Z}^T \cdot \mathbf{k}. \tag{86}$$

There exists the obvious identity

$$(\mathbf{E} - \mathbf{Q}(\theta)) \cdot \theta = \left(\mathbf{E} - \mathbf{Q}^{\mathsf{T}}\right) \cdot \theta = \mathbf{0} \Longrightarrow (\mathbf{a} - \mathbf{a}') \cdot \theta = 0$$

for arbitrary $\mathbf{a}$, $\mathbf{a}' = \mathbf{Q} \cdot \mathbf{a}$. Taking into account this identity, we may obtain

$$\left(\mathbf{E} - \mathbf{Q}^{\mathsf{T}}(\theta)\right) \cdot \mathbf{M} = \frac{\partial F}{\partial (\mathbf{k} \cdot \theta)} \mathbf{k} \times \theta.$$

Multiplying this equality by the vector $\mathbf{k}$ we obtain

$$(\mathbf{k} - \mathbf{k}') \cdot \mathbf{M} = 0. \tag{87}$$

For the isotropic potential, equality (87) holds for any vector $\mathbf{a}$. Sometimes equality (87) is very important.

11.2 The equations of motion of a rigid body on elastic foundation

The inertia tensor is supposed to transversally isotropic

$$\mathbf{A} = A_1 (\mathbf{E} - \mathbf{k} \otimes \mathbf{k}) + A_3 \mathbf{k} \otimes \mathbf{k}, \quad \mathbf{d}_3 = \mathbf{k}, \qquad A_1 = A_2. \tag{88}$$

The position of a body at the instant t is called the actual position of a body. The motion of the body can be defined either by the turn-tensor $\mathbf{P}(t)$ or by the turn-vector $\theta(t)$

$$\mathbf{P}(t) = \mathbf{Q}(\theta(t)).$$

The tensor of inertia $\mathbf{A}^{(t)}$ in the actual position is determined by

$$\mathbf{A}^{(t)} = \mathbf{P}(t) \cdot \mathbf{A} \cdot \mathbf{P}^{\mathsf{T}}(t). \tag{89}$$

If the tensor $\mathbf{A}$ is transversally isotropic, this results in

$$\mathbf{A}^{(t)} = A_1 (\mathbf{E} - \mathbf{k}' \otimes \mathbf{k}') + A_3 \mathbf{k}' \otimes \mathbf{k}', \qquad \mathbf{k}' = \mathbf{P} \cdot \mathbf{k}. \tag{90}$$

The kinetic moment of the body can be expressed in two forms. In terms of the left angular velocity we obtain

$$\mathbf{L} = \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{P}^{\mathsf{T}} \cdot \boldsymbol{\omega} = A_1 \boldsymbol{\omega} + (A_3 - A_1)(\mathbf{k}' \cdot \boldsymbol{\omega}) \mathbf{k}'. \tag{91}$$

In terms of the right angular velocity the kinetic moment has the form

$$\mathbf{L} = \mathbf{P} \cdot \mathbf{A} \cdot \boldsymbol{\Omega} = \mathbf{P} \cdot [A_1 \boldsymbol{\Omega} + (A_3 - A_1)(\mathbf{k} \cdot \boldsymbol{\Omega}) \mathbf{k}]. \tag{92}$$

Let us note that

$$\mathbf{k}' \cdot \boldsymbol{\omega} = \mathbf{k} \cdot \mathbf{P}^{\mathsf{T}} \cdot \boldsymbol{\omega} = \mathbf{k} \cdot \boldsymbol{\Omega}. \tag{93}$$

An external moment $\mathbf{M}$ acting on the body can be represented in the form

$$\mathbf{M} = \mathbf{M}_e + \mathbf{M}_{ext},$$

where $\mathbf{M}_e$ is a reaction of the elastic foundation and $\mathbf{M}_{ext}$ is an additional external moment. The elastic moment $\mathbf{M}_e$ is supposed to be positional one

$$\mathbf{M}_e = -\mathbf{Z}^T(\theta) \cdot \frac{dU(\theta)}{d\theta}. \tag{94}$$

The scalar function $U(\theta)$ is called the elastic energy. In the following, the elastic foundation is supposed to be transversally isotropic. Then the elastic moment can be represented in form (86), i.e.

$$\mathbf{M}_e(\theta) = -C\left(\theta^2, \mathbf{k}\cdot\theta\right)\theta - D\left(\theta^2, \mathbf{k}\cdot\theta\right)\mathbf{Z}^T(\theta)\cdot\mathbf{k}, \tag{95}$$

where the unit vector $\mathbf{k}$ is placed on the axis of isotropy of the body in the reference position, and

$$C = 2\frac{\partial}{\partial(\theta^2)}U\left(\theta^2, \mathbf{k}\cdot\theta\right), \qquad D = \frac{\partial}{\partial(\mathbf{k}\cdot\theta)}U\left(\theta^2, \mathbf{k}\cdot\theta\right). \tag{96}$$

For an external moment $\mathbf{M}_{ext}$ let us accept the expression

$$\mathbf{M}_{ext} = -\mathbf{Z}^T(\theta)\cdot\frac{dV(\theta)}{d\theta} + \mathbf{M}_{ex}, \tag{97}$$

where the first term describes the potential part of the external moment.

The second law of dynamics by Euler can be represented in two equivalent forms. In terms of the left angular velocity we find from $\dot{\mathbf{L}} = \mathbf{M}$

$$\left[\mathbf{P}(\theta)\cdot\mathbf{A}\cdot\mathbf{P}^T(\theta)\cdot\omega\right]^{\cdot} + \mathbf{Z}^T(\theta)\cdot\frac{d(U+V)}{d\theta} = \mathbf{M}_{ex}. \tag{98}$$

This equation has to be completed by the left Poisson equations

$$\dot{\theta} = \omega - \frac{1}{2}\theta\times\omega + \frac{1-g}{\theta^2}\theta\times(\theta\times\omega). \tag{99}$$

In terms of the right angular velocity, the model (98)–(99) can be represented as

$$\mathbf{A}\cdot\dot{\mathbf{\Omega}} + \mathbf{\Omega}\times\mathbf{A}\cdot\mathbf{\Omega} + \mathbf{Z}(\theta)\cdot\frac{d(U+V)}{d\theta} = \mathbf{P}^T(\theta)\cdot\mathbf{M}_{ex}, \tag{100}$$

$$\dot{\theta} = \mathbf{\Omega} + \frac{1}{2}\theta\times\mathbf{\Omega} + \frac{1-g}{\theta^2}\theta\times(\theta\times\mathbf{\Omega}). \tag{101}$$

11.3 The regular precession

Let us consider a body with a transversally isotropic tensor of inertia. The elastic foundation is supposed to be transversally isotropic as well. The equations of motion are given by expressions (98), (99) and expression (95) for the elastic moment:

$$[A_1\omega + (A_3 - A_1)(\mathbf{k}'\cdot\omega)\mathbf{k}']^{\cdot} + C\theta + D\mathbf{Z}^T\cdot\mathbf{k} = \mathbf{0}, \quad \mathbf{k}' = \mathbf{P}\cdot\mathbf{k}, \tag{102}$$

$$\dot{\theta} = \omega - \frac{1}{2}\theta\times\omega + \frac{1-g}{\theta^2}\theta\times(\theta\times\omega), \tag{103}$$

where the functions C and D are defined by (96). We assume a particular solution of system (102), (103) to be represented in the form

$$\theta = \vartheta \mathbf{p}', \quad \mathbf{p}' = \mathbf{Q}(\psi \mathbf{k}) \cdot \mathbf{p}, \quad \mathbf{p} \cdot \mathbf{k} = 0, \quad \mathbf{P} = \mathbf{Q}(\vartheta \mathbf{p}'), \tag{104}$$

where the motion (104) is called a regular precession if

$$\vartheta = \text{const}, \quad \dot{\psi} = \text{const} \quad \Rightarrow \quad \dot{\boldsymbol{\theta}} = \dot{\psi} \mathbf{k} \times \boldsymbol{\theta}. \tag{105}$$

The left angular velocity is given as

$$\boldsymbol{\omega} = \mathbf{Q}(\psi \mathbf{k}) \cdot \boldsymbol{\omega}_0, \quad \boldsymbol{\omega}_0 = \dot{\psi} \left[(1 - \cos\vartheta) \mathbf{k} + \sin\vartheta \mathbf{k} \times \mathbf{p} \right] = \text{const}. \tag{106}$$

We see that the angular velocity vector ω is a precession of the vector ω_0 around the axis $\mathbf{k}$ orthogonal to the turn-vector:

$$\boldsymbol{\theta} \cdot \boldsymbol{\omega} = \boldsymbol{\theta} \cdot \boldsymbol{\Omega} = 0, \quad \mathbf{k} \cdot \boldsymbol{\theta} = 0.$$

In addition, let us accept the restriction

$$\mathsf{D}\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right) |_{\mathbf{k} \cdot \boldsymbol{\theta} = 0} = \frac{\partial}{\partial (\mathbf{k} \cdot \boldsymbol{\theta})} \mathsf{U}\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right) |_{\mathbf{k} \cdot \boldsymbol{\theta} = 0} = 0,$$

which is satisfied for most kinds of elastic energy. Then we obtain from Eq. (102) for the assumed solution

$$\dot{\psi}^2 = \frac{\mathsf{C}\left(\vartheta^2, 0\right) \vartheta}{\sin\vartheta \left[A_3 (1 - \cos\vartheta) + A_1 \cos\vartheta\right]}. \tag{107}$$

11.4 The inertia elastic foundation.

Let us consider the inertia elastic foundation. In such a case instead of (95) we shall get

$$\mathbf{M}_e(\boldsymbol{\theta}) = -\mathsf{C}\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right) \boldsymbol{\theta} - \mathsf{D}\left(\theta^2, \mathbf{k} \cdot \boldsymbol{\theta}\right) \mathbf{Z}^{\mathsf{T}}(\boldsymbol{\theta}) \cdot \mathbf{k} - \mu \boldsymbol{\theta} \times \boldsymbol{\omega}. \tag{108}$$

Instead of (102)–(103) we obtain

$$\left[A_1 \boldsymbol{\omega} + (A_3 - A_1)(\mathbf{k}' \cdot \boldsymbol{\omega}) \mathbf{k}'\right]^{\cdot} + \mathsf{C} \boldsymbol{\theta} + \mathsf{D} \mathbf{Z}^{\mathsf{T}} \cdot \mathbf{k} + \mu \boldsymbol{\theta} \times \boldsymbol{\omega} = \mathbf{0}, \tag{109}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \frac{1}{2} \boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1 - g}{\theta^2} \boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega}), \quad \mathbf{k}' = \mathbf{P} \cdot \mathbf{k}. \tag{110}$$

It is to show that the regular precession is impossible. The angle of nutation tends to zero and the Nikolai paradox is impossible.

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A Micro-Polar Theory for Piezoelectric Materials*

Abstract

Theory of the piezoelectric materials had been developed many years ago. There exist a several theories of the piezoelectricity. All of them lead to the very complicated equations. The exact solutions of these equations may be found only for very particular cases. By this reason it is not easy to compare theoretical and experimental results. At the present time it seems to be possible to say that there is no qualitative discrepancies between theory and experiments. From the pure theoretical point of view in the theory of the piezoelectricity there are some serious problems. It was supposed that the stress state of the piezoelectric material can be described by means of the symmetrical stress tensor. However some piezoelectric materials are the dipole crystals. In such a case the rotation degrees of freedom must be taken into account. It means that the theory of the piezoelectric materials must be constructed on the base of the micro-polar continuum. The theory of such a kind is presented in the report. The basic equations are derived from the fundamental laws of Eulerian mechanics and contain two unsymmetrical stress tensors. The theory presented in the report differs from conventional theory very significantly. However under some assumptions this theory may be reduced to the classical one. The theory was tested on some simple problems and results were compared with classical ones.

1 Classical set of equations

Here we briefly present main equations of the classical theory.

Equation of motion:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \mathbf{F} = \rho \ddot{\mathbf{u}}, \tag{1}$$

where $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathsf{T}}$ is the symmetric stress tensor, ρ is the mass density, $\mathbf{u}$ is the displacement vector.

Poisson equation:

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 0, \quad \boldsymbol{\nabla} \cdot \mathbf{E}' = 0, \tag{2}$$

where $\mathbf{D} = \mathbf{E}' + 4\pi\mathbf{P}$ is the vector of electric displacement density, $\mathbf{E}'$ is the vector of electric field intensity in vacuum, $\mathbf{P}$ is the vector of density of induced polarization.

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The energy balance equation in classics has the following form:

$$\rho\dot{\mathbb{U}} = \boldsymbol{\tau}\cdot\cdot\dot{\varepsilon} + \mathbf{E}\cdot\dot{\mathbf{D}}, \tag{3}$$

The electric enthalpy density is expressed by

$$\mathbb{F} = \mathbb{U} - \mathbf{E}\cdot\mathbf{D}. \tag{4}$$

For the linear approximation the following biquadratic form is employed:

$$\rho\mathbb{F} = \rho\mathbb{F}_0 + \frac{1}{2}\varepsilon\cdot\cdot\mathbf{C}\cdot\cdot\varepsilon - \mathbf{E}\cdot\mathbf{M}\cdot\cdot\varepsilon - \frac{1}{2}\mathbf{E}\cdot\boldsymbol{\epsilon}\cdot\mathbf{E}. \tag{5}$$

Constitutive equations are expressed by:

$$\boldsymbol{\tau} = \mathbf{C}\cdot\cdot\varepsilon - \mathbf{E}\cdot\mathbf{M}, \tag{6}$$

$$\mathbf{D} = \mathbf{M}\cdot\cdot\varepsilon + \boldsymbol{\epsilon}\cdot\mathbf{E}, \tag{7}$$

where $\mathbf{E}$ is the vector electric field intensity in medium, ε is the tensor of linear deformation, $\mathbf{C}$ is the tensor of elasticity, $\mathbf{M}$ is the tensor of piezoelectricity, $\boldsymbol{\epsilon}$ is the tensor of dielectric permittivity.

This notation uses displacements and electric field as independent variables. The other notation assumes, that independent variables are displacements and electric displacement. This notation of classical theory uses the intrinsic energy instead of the electric enthalpy. When equation

$$\rho\mathbb{U} = \rho\mathbb{U}_0 + \frac{1}{2}\varepsilon\cdot\cdot\mathbf{C}^{(c)}\cdot\cdot\varepsilon + \mathbf{D}\cdot\mathbf{M}^{(c)}\cdot\cdot\varepsilon + \frac{1}{2}\mathbf{D}\cdot\boldsymbol{\epsilon}^{(c)}\cdot\mathbf{D}, \tag{8}$$

is used, then the constitutive equations are expressed by equations:

$$\boldsymbol{\tau} = \mathbf{C}^{(c)}\cdot\cdot\varepsilon + \mathbf{D}\cdot\mathbf{M}^{(c)}, \tag{9}$$

$$\mathbf{E} = \mathbf{M}^{(c)}\cdot\cdot\varepsilon + \boldsymbol{\epsilon}^{(c)}\cdot\mathbf{D}. \tag{10}$$

2 Particle model

In the present work we present a micro-polar theory for mediums with non-zero electric dipole momentum density. Such a materials possesses piezoelectric properties, i.e. the electric field influence on the mechanical state of medium. In order to write equations for piezoelectric media, we have presented the particle model of such media. We use lagrangian description.

Let the particles be point bodies with dipole properties. The particles have abilities to move in space and rotate. Let the particle have the ability to change the value of dipole as well, i.e. the particle is elastic point-body. Let the particle with dipole value $\mathbf{d}_0$ in the reference configuration be characterized by following parameters: $\mathbf{R}_0^+$ and $\mathbf{R}_0^-$ are the vectors of charges q^+ and q^-, respectively ($q^+ = -q^- = q$ are the charge values); $\mathbf{r}_0$ is the radius of geometrical center of the particle.

Let in the actual configuration charges $q^\pm$ be moved from points $\mathbf{R}_0^\pm$ to points $\mathbf{R}^\pm$. Respectively, the center of particle is moved to the point $\mathbf{r}$. Let us introduce the following notations:

$$\mathbf{u} = \mathbf{r} - \mathbf{r}_0, \tag{11}$$

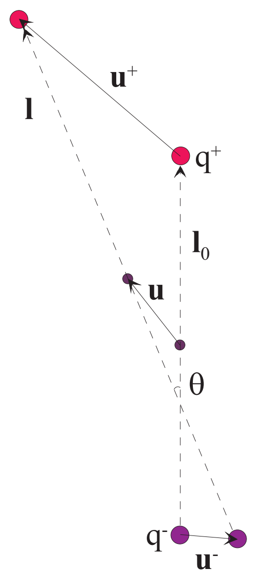

Figure 1: Point-body as a couple of charges

$$\mathbf{u}^{+} = \mathbf{R}^{+} - \mathbf{R}_0^{+}, \tag{12}$$

$$\mathbf{u}^{-} = \mathbf{R}^{-} - \mathbf{R}_0^{-}. \tag{13}$$

$$\mathbf{d}_0 = q\mathbf{l}_0 = q(\mathbf{R}_0^{+} - \mathbf{R}_0^{-}), \tag{14}$$

$$\mathbf{d} = q\mathbf{l} = q(\mathbf{R}^{+} - \mathbf{R}^{-}), \tag{15}$$

Let $\boldsymbol{\theta}(|\boldsymbol{\theta}| \ll 1)$ be the turn of the vector $\mathbf{d}_0$ to $\mathbf{d}$. Let $\mathbf{p} \equiv \mathbf{d} - \mathbf{d}_0$ be the change in dipole state. Let δ be the relative dipole value change

$$|\mathbf{d}| = |\mathbf{d}_0|(1+\delta). \tag{16}$$

After certain transformations for $\mathbf{p}$ we have decomposition

$$\mathbf{p} \simeq \mathbf{p}_1 + \mathbf{p}_2, \tag{17}$$

where

$$\mathbf{p}_1 = \delta\mathbf{d}_0, \qquad \mathbf{p}_2 = (1+\delta)\boldsymbol{\theta} \times \mathbf{d}_0. \tag{18}$$

Let us define the piezoelectric polarization density of continuum as

$$\boldsymbol{\mathcal{P}}^p = \lim_{\Delta V \to 0} \frac{\sum_{k \in \Delta V}(\mathbf{p}_{1k} + \mathbf{p}_{2k})}{\Delta V} = \boldsymbol{\mathcal{P}}_1^p + \boldsymbol{\mathcal{P}}_2^p, \tag{19}$$

where

$$\boldsymbol{\mathcal{P}}_1^p = \delta\boldsymbol{\mathcal{P}}^s, \qquad \boldsymbol{\mathcal{P}}_2^p = (1+\delta)\boldsymbol{\theta} \times \boldsymbol{\mathcal{P}}^s, \tag{20}$$

where $\boldsymbol{\mathcal{P}}^s$ is the density of spontaneous polarization.

The polarization vector $\mathcal{P}^p$ is the sum of two orthogonal polarization vectors of different nature: the first one is associated with rotation of medium particle, the second one is associated with changing of its absolute value.

Let us now write the power, given to the point-body by effective electric field. For this purpose we use the Lorentz formulae and Poisson equation. Finally, after transformations we have:

$$\dot{\mathbf{e}} = (\nabla \cdot \mathbf{E})\mathbf{d}_0 \cdot \dot{\mathbf{u}} + \mathbf{E} \cdot \dot{\mathbf{p}}. \tag{21}$$

Using equations (18), let us write the time derivative of the vector (17):

$$\dot{\mathbf{p}} = \dot{\delta}(\mathbf{d}_0 + \theta \times \mathbf{d}_0) + (1+\delta)\dot{\theta} \times \mathbf{d}_0 \simeq \dot{\delta}\mathbf{d}_0 + \dot{\theta} \times \mathbf{d}_0. \tag{22}$$

Now, the equation for medium may be written:

$$\dot{\mathcal{E}} = (\nabla \cdot \mathbf{E})\mathcal{P}^s \cdot \dot{\mathbf{u}} + (1+\delta)(\mathcal{P}^s \times \mathbf{E}) \cdot \dot{\theta} + (\mathcal{P}^s \cdot \mathbf{E})\dot{\delta}. \tag{23}$$

3 The Laws of motion

Kinetic energy for the point body with inertia tensor $\mathbf{J}$ is represented in the form:

$$\mathcal{K} = \frac{1}{2}m\dot{\mathbf{u}}^2 + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{Q}(\mathbf{t}) \cdot \mathbf{J} \cdot \mathbf{Q}(\mathbf{t})^T \cdot \boldsymbol{\omega}. \tag{24}$$

The density of momentum:

$$\mathbf{K}_1 = \frac{\partial \mathcal{K}}{\partial \dot{\mathbf{u}}} = \rho\dot{\mathbf{u}},$$

where $\rho = V^{-1} \sum_V m_i$ is the mass density, V is material volume. The equation of momentum balance is expressed by

$$\frac{d}{dt}\int_V \mathbf{K}_1 \, dV = \frac{d}{dt}\int_V \rho\dot{\mathbf{u}} dV = \int_V \rho\mathbb{F} dV + \int_S \mathbf{T}_{(n)} dS, \tag{25}$$

where $\mathbb{F}$ is external force density, $\mathbf{T}_{(n)}$ is stress vector. The following formulae is true:

$$\mathbf{T}_{(n)} = \mathbf{n} \cdot \mathbf{T}, \tag{26}$$

where $\mathbf{n}$ is normal vector, $\mathbf{T}$ is Cauchy stress tensor. Let us apply Green theorem

$$\int_S \mathbf{T}_{(n)} d\mathbf{S} = \int_S \mathbf{n} \cdot \mathbf{T} d\mathbf{S} = \int_V \nabla \cdot \mathbf{T} dV$$

and, taking into account (26), the equation (25) become

$$\int_V [\rho\ddot{\mathbf{u}} - \rho\mathbb{F} - \nabla \cdot \mathbf{T}] \, dV = 0. \tag{27}$$

The momentum balance equation:

$$\nabla \cdot \mathbf{T} + \rho\mathbb{F} = \rho\ddot{\mathbf{u}}. \tag{28}$$

The other form of this equation is expressed by

$$\nabla \cdot \boldsymbol{\tau} - \frac{1}{2}\nabla \times \mathbf{q} + \rho\mathbb{F} = \rho\ddot{\mathbf{u}}, \tag{29}$$

where $\boldsymbol{\tau} = \frac{1}{2}(\mathbf{T} + \mathbf{T}^{\mathsf{T}})$, and $\mathbf{q} = \mathbf{T}_{\times}$.

For turn-tensor $\mathbf{Q}$ it is possible to write the following approximation:

$$\mathbf{Q} \approx \mathbf{I} + \boldsymbol{\phi} \times \mathbf{I}, \qquad \boldsymbol{\phi} = \phi\mathbf{e}_{\phi}. \tag{30}$$

Using Poisson equation

$$\dot{\mathbf{Q}} = \boldsymbol{\omega} \times \mathbf{Q}, \tag{31}$$

the expression for angular velocity follows:

$$\boldsymbol{\omega} = \dot{\boldsymbol{\phi}}.$$

For small $\boldsymbol{\omega}$, the equation for kinetic momentum is expressed by:

$$\mathbf{K}_2 = \mathbf{r} \times \mathbf{K}_1 + \frac{\partial \mathcal{K}}{\partial \dot{\boldsymbol{\phi}}} = \rho(\mathbf{r} \times \dot{\mathbf{u}} + \mathbf{J} \cdot \dot{\boldsymbol{\phi}}(x, t)). \tag{32}$$

Integral form of the second Law of dynamics is as follows:

$$\frac{d}{dt}\int_V \mathbf{K}_2 dV = \int_V \rho\left(\mathbf{r} \times \mathbb{F} + \mathbb{L}\right) dV + \int_S \left(\mathbf{r} \times \mathbf{T}_{(n)} + \boldsymbol{\mu}_{(n)}\right) dS, \tag{33}$$

where $\mathbb{L}$ is the external momentum, $\boldsymbol{\mu}_{(n)}$ is the momentum stress tensor. The Cauchy equation for $\boldsymbol{\mu}$ is:

$$\boldsymbol{\mu}_{(n)} = \mathbf{n} \cdot \boldsymbol{\mu}. \tag{34}$$

The local form of the second Law is expressed by:

$$\nabla \cdot \boldsymbol{\mu} + \mathbf{q} + \rho\mathbb{L} = \rho\mathbf{J} \cdot \ddot{\boldsymbol{\phi}}. \tag{35}$$

Let us suppose

$$\boldsymbol{\mu} = \mathbf{m} \times \mathbf{I}. \tag{36}$$

This is not the only available representation, but it is not so important today to specify it more precisely. Finally we have the following equation:

$$\nabla \times \mathbf{m} + \mathbf{q} + \rho\mathbb{L} = \rho\mathbf{J} \cdot \ddot{\boldsymbol{\phi}}. \tag{37}$$

The integral form of the energy balance equation is expressed by:

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\rho\dot{\mathbf{u}}^2 + \frac{1}{2}\rho\dot{\boldsymbol{\phi}} \cdot \mathbf{J} \cdot \dot{\boldsymbol{\phi}} + \rho\mathbb{U}\right) dV = \int_V \left(\rho\mathbb{F} \cdot \dot{\mathbf{u}} + \rho\mathbb{L} \cdot \dot{\boldsymbol{\phi}} + Q\right) dV + \\ + \int_S \left(\mathbf{T}_{(n)} \cdot \dot{\mathbf{u}} + \boldsymbol{\mu}_{(n)} \cdot \dot{\boldsymbol{\phi}} + \mathbf{H} \cdot \mathbf{n}\right) dS, \tag{38}$$

where $\mathbf{H}$ is energy flow vector, Q is density of external energy supply sources. Generally, Q represent the energy dissipation.

The local form of energy balance equation is represented as follows:

$$\rho\dot{\mathbb{U}} = \boldsymbol{\tau} \cdot\cdot\, \dot{\boldsymbol{\varepsilon}} - \mathbf{q} \cdot \dot{\boldsymbol{\theta}} - \mathbf{m} \cdot \dot{\boldsymbol{\gamma}} + \nabla \cdot \mathbf{H} + Q. \tag{39}$$

where

$$\boldsymbol{\varepsilon} \equiv \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathsf{T}}\right), \quad \boldsymbol{\theta} \equiv \boldsymbol{\phi} - \frac{1}{2}\nabla \times \mathbf{u}, \quad \boldsymbol{\gamma} = \nabla \times \boldsymbol{\phi}. \tag{40}$$

4 Equations of piezoelectric medium.

Let electric field be the only external influence. Then, from equation (23), we can obtain:

$$\rho\mathbb{F} = (\boldsymbol{\nabla} \cdot \mathbf{E})\boldsymbol{\mathcal{P}}^s, \tag{41}$$

$$\rho\mathbb{L} = (1 + \delta)\boldsymbol{\mathcal{P}}^s \times \mathbf{E}. \tag{42}$$

The last term in equation (23) comes directly to the intrinsic energy equation.

$$Q = (\boldsymbol{\mathcal{P}}^s \cdot \mathbf{E})\dot{\delta}. \tag{43}$$

Let us introduce two scalars: the temperature T and the entropy S. Let that values satisfy the following equation:

$$T\dot{S} = \boldsymbol{\nabla} \cdot \mathbf{H} + Q_i, \tag{44}$$

where Q_i represent the work of dissipative forces

$$Q_i = \boldsymbol{\tau}_i(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}) \cdot\cdot \dot{\boldsymbol{\varepsilon}} - \mathbf{q}_i(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \cdot \dot{\boldsymbol{\theta}} - \mathbf{m}_i(\boldsymbol{\gamma}, \dot{\boldsymbol{\gamma}}) \cdot \dot{\boldsymbol{\gamma}}. \tag{45}$$

For heat flow vector **H** it is possible to write equation:

$$\mathbf{H} = -\chi\boldsymbol{\nabla}T. \tag{46}$$

The equation for intrinsic energy may be rewritten:

$$\rho\dot{\mathbb{U}} = \boldsymbol{\tau} \cdot\cdot \dot{\boldsymbol{\varepsilon}} - \mathbf{q} \cdot \dot{\boldsymbol{\theta}} - \mathbf{m} \cdot \dot{\boldsymbol{\gamma}} + (\mathbf{E} \cdot \boldsymbol{\mathcal{P}}^s)\dot{\delta} + T\dot{S}. \tag{47}$$

Let us assume the hypothesis of natural state and represent the intrinsic energy as positively defined bilinear quadratic form:

$$\begin{aligned}\rho\mathbb{U} = \rho\mathbb{U}_0 + \frac{1}{2}\boldsymbol{\varepsilon} \cdot\cdot \mathbf{C}^{(\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\theta} \cdot \mathbf{C}^{(\theta)} \cdot \boldsymbol{\theta} + \frac{1}{2}\boldsymbol{\gamma} \cdot \mathbf{C}^{(\gamma)} \cdot \boldsymbol{\gamma} + \frac{1}{2}C^{(\delta)}\delta^2 + \frac{1}{2}C^{(S)}S^2 + \\ + \boldsymbol{\theta} \cdot \mathbf{C}^{(\theta\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + \boldsymbol{\gamma} \cdot \mathbf{C}^{(\gamma\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + \delta\mathbf{C}^{(\delta\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + S\mathbf{C}^{(S\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + \\ + \boldsymbol{\gamma} \cdot \mathbf{C}^{(\gamma\theta)} \cdot \boldsymbol{\theta} + \delta\mathbf{C}^{(\delta\theta)} \cdot \boldsymbol{\theta} + S\mathbf{C}^{(S\theta)} \cdot \boldsymbol{\theta} + \\ + \delta\mathbf{C}^{(\delta\gamma)} \cdot \boldsymbol{\gamma} + S\mathbf{C}^{(S\gamma)} \cdot \boldsymbol{\gamma} + C^{(\delta S)}\delta S.\end{aligned} \tag{48}$$

Then, it is possible to write Cauchy-Green relations:

$$\boldsymbol{\tau} = \frac{\partial\rho\mathbb{U}}{\partial\boldsymbol{\varepsilon}} = \mathbf{C}^{(\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + \boldsymbol{\theta} \cdot \mathbf{C}^{(\theta\varepsilon)} + \boldsymbol{\gamma} \cdot \mathbf{C}^{(\gamma\varepsilon)} + \mathbf{C}^{(\delta\varepsilon)}\delta + \mathbf{C}^{(S\varepsilon)}S, \tag{49}$$

$$-\mathbf{q} = \frac{\partial\rho\mathbb{U}}{\partial\boldsymbol{\theta}} = \mathbf{C}^{(\theta\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + \mathbf{C}^{(\theta)} \cdot \boldsymbol{\theta} + \boldsymbol{\gamma} \cdot \mathbf{C}^{(\gamma\theta)} + \mathbf{C}^{(\delta\theta)}\delta + \mathbf{C}^{(S\theta)}S, \tag{50}$$

$$-\mathbf{m} = \frac{\partial\rho\mathbb{U}}{\partial\boldsymbol{\gamma}} = \mathbf{C}^{(\gamma\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + \mathbf{C}^{(\gamma\theta)} \cdot \boldsymbol{\theta} + \mathbf{C}^{(\gamma)} \cdot \boldsymbol{\gamma} + \mathbf{C}^{(\delta\gamma)}\delta + \mathbf{C}^{(S\gamma)}S, \tag{51}$$

$$\mathbf{E} \cdot \boldsymbol{\mathcal{P}}^s = \frac{\partial\rho\mathbb{U}}{\partial\delta} = \mathbf{C}^{(\delta\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + \mathbf{C}^{(\delta\theta)} \cdot \boldsymbol{\theta} + \mathbf{C}^{(\delta\gamma)} \cdot \boldsymbol{\gamma} + C^{(\delta)}\delta + C^{(\delta S)}S, \tag{52}$$

$$T = \frac{\partial \rho \mathbb{U}}{\partial S} = \mathbf{C}^{(S\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + \mathbf{C}^{(S\theta)} \cdot \boldsymbol{\theta} + \mathbf{C}^{(S\gamma)} \cdot \boldsymbol{\gamma} + C^{(\delta S)} \delta + C^{(S)} S. \tag{53}$$

The equation of motion are as follows:

$$\nabla \cdot \boldsymbol{\tau} - \frac{1}{2} \nabla \times \mathbf{q} + (\nabla \cdot \mathbf{E}) \boldsymbol{\mathcal{P}}^s = \rho \ddot{\mathbf{u}}, \tag{54}$$

$$\nabla \times \mathbf{m} + \mathbf{q} + (1 + \delta) \boldsymbol{\mathcal{P}}^s \times \mathbf{E} = \rho \mathbf{J} \cdot \ddot{\boldsymbol{\phi}}, \tag{55}$$

$$-\chi \nabla^2 T + Q_i = T\dot{S}. \tag{56}$$

5 Micropolar theory: transformation to classical form

In order to compare the set of equations described above with equations of classical theory, let us try to make some assumptions and simplify the micropolar theory. Let us rewrite equation (19), taking into account (20), and suppose $\delta \ll 1$:

$$\boldsymbol{\mathcal{P}}^p = \delta \boldsymbol{\mathcal{P}}^s + \boldsymbol{\theta} \times \boldsymbol{\mathcal{P}}^s. \tag{57}$$

From this equation it is obvious, that for known δ and $\boldsymbol{\theta}$ it is possible to find $\boldsymbol{\mathcal{P}}^p$. Reverse task has no single solution. Let us suppose $\boldsymbol{\theta} \cdot \boldsymbol{\mathcal{P}}^s = 0$. Then, from (57) it is possible to write:

$$\boldsymbol{\theta} = \frac{\boldsymbol{\mathcal{P}}^s \times \boldsymbol{\mathcal{P}}^p}{|\boldsymbol{\mathcal{P}}^s|^2} = \boldsymbol{\chi}^s \times \boldsymbol{\mathcal{P}}^p, \tag{58}$$

$$\delta = \frac{\boldsymbol{\mathcal{P}}^s \cdot \boldsymbol{\mathcal{P}}^p}{|\boldsymbol{\mathcal{P}}^s|^2} = \boldsymbol{\chi}^s \cdot \boldsymbol{\mathcal{P}}^p. \tag{59}$$

where $\boldsymbol{\chi}^s = \boldsymbol{\mathcal{P}}^s / |\boldsymbol{\mathcal{P}}^s|^2$. Assume, that $\boldsymbol{\phi} = 0$ (this assumption is valid for crystalline structures) and neglect the temperature effects. Equation (55) rewrite as follows:

$$\mathbf{q} = -\boldsymbol{\mathcal{P}}^s \times \mathbf{E}. \tag{60}$$

Then, relations (49)–(52) become:

$$\boldsymbol{\tau} = \mathbf{C}^{(\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + \boldsymbol{\mathcal{P}}^p \cdot (-\boldsymbol{\chi}^s \times \mathbf{C}^{(\theta\varepsilon)} + \boldsymbol{\chi}^s \otimes \mathbf{C}^{(\delta\varepsilon)}), \tag{61}$$

$$\boldsymbol{\mathcal{P}}^s \times \mathbf{E} = \mathbf{C}^{(\theta\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + (\mathbf{C}^{(\theta)} \times \boldsymbol{\chi}^s + \mathbf{C}^{(\delta\theta)} \otimes \boldsymbol{\chi}^s) \cdot \boldsymbol{\mathcal{P}}^p, \tag{62}$$

$$\boldsymbol{\mathcal{P}}^s \cdot \mathbf{E} = \mathbf{C}^{(\delta\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + (\mathbf{C}^{(\delta\theta)} \times \boldsymbol{\chi}^s + \mathbf{C}^{(\delta)} \otimes \boldsymbol{\chi}^s) \cdot \boldsymbol{\mathcal{P}}^p, \tag{63}$$

From equations (62) and (63) it follows the expression for $\mathbf{E}$:

$$\begin{aligned} \mathbf{E} = {} & (-\boldsymbol{\chi}^s \times \mathbf{C}^{(\theta\varepsilon)} + \boldsymbol{\chi}^s \otimes \mathbf{C}^{(\delta\varepsilon)}) \cdot\cdot \boldsymbol{\varepsilon} + \\ & + (-\boldsymbol{\chi}^s \times \mathbf{C}^{(\theta)} \times \boldsymbol{\chi}^s + \boldsymbol{\chi}^s \otimes \mathbf{C}^{(\delta\theta)} \times \boldsymbol{\chi}^s - \boldsymbol{\chi}^s \times \mathbf{C}^{(\delta\theta)} \otimes \boldsymbol{\chi}^s + \mathbf{C}^{(\delta)} \boldsymbol{\chi}^s \otimes \boldsymbol{\chi}^s) \cdot \boldsymbol{\mathcal{P}}^p. \end{aligned}$$

Terms $\chi^s \otimes \mathbf{C}^{(\delta\theta)} \times \chi^s = 0$ and $\chi^s \times \mathbf{C}^{(\delta\theta)} \otimes \chi^s = 0$. This is obvious from the symmetry of the system: either the vectors χ^s and $\mathbf{C}^{(\delta\theta)}$ must be collinear at the given point of medium or vector $\mathbf{C}^{(\delta\theta)} = 0$ due to structure's symmetry.

Thus, we obtain the system of equations

$$\boldsymbol{\tau} = \mathbf{C}^{(n)} \cdot\cdot \varepsilon + \mathcal{P}^p \cdot \mathbf{M}^{(n)}, \tag{64}$$

$$\mathbf{E} = \mathbf{M}^{(n)} \cdot\cdot \varepsilon + \epsilon^{(n)} \cdot \mathcal{P}^p. \tag{65}$$

The following notations are used:

$$\mathbf{C}^{(n)} = \mathbf{C}^{(\varepsilon)}, \tag{66}$$

$$\mathbf{M}^{(n)} = \chi^s \otimes \mathbf{C}^{(\delta\varepsilon)} - \chi^s \times \mathbf{C}^{(\theta\varepsilon)}, \tag{67}$$

$$\epsilon^{(n)} = \mathbf{C}^{(\delta)} \chi^s \otimes \chi^s - \chi^s \times \mathbf{C}^{(\theta)} \times \chi^s. \tag{68}$$

The obtained equations (64)–(65) have similar shape compared to equations of classical theory (9)–(10). Moreover, comparison of the shape of mentioned material tensor with material tensors of classical theory shows the identical non-zero components placement, i.e. these material tensors are equivalent.

6 The system of equations for two-dimensional layer.

Let us consider the layer, made from material with two orthogonal planes of symmetry. Let thickness be along $\mathbf{e}_3$ direction. Then, for displacements we have equations:

$$\mathbf{u} = u_1(x_1, x_3)\mathbf{e}_1 + u_3(x_1, x_3)\mathbf{e}_3, \qquad \boldsymbol{\phi} = \phi_2(x_1, x_3)\mathbf{e}_2. \tag{69}$$

Let the spontaneous polarization and electric field be along the $\mathbf{e}_3$ axis:

$$\mathbf{P}^{(s)} = P^{(s)}\mathbf{e}_3, \qquad \mathbf{E} = E_3(x_1, x_3)\mathbf{e}_3. \tag{70}$$

The form of material tensors must be obtained, using the theory of symmetry. In order to simplify equations, here and further we will express δ from equation (52) and substitute it into equations (49), (50) and (51). Also, we will neglect temperature effects for the same reason. After transformations, the following system of 3 differential equations of the second order is obtained:

$$C_{11}\frac{\partial^2 u_1}{\partial x_1^2} + C_{12}\frac{\partial^2 u_1}{\partial x_3^2} + C_{13}\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + C_{14}\frac{\partial^2 \phi_2}{\partial x_1^2} + C_{15}\frac{\partial \phi_2}{\partial x_3} + C_{16}\frac{\partial^2 \phi_2}{\partial x_3^2} + \\ + C_{17}P^{(s)}\frac{\partial E_3}{\partial x_1} = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{71}$$

$$C_{21}\frac{\partial^2 u_3}{\partial x_1^2} + C_{22}\frac{\partial^2 u_3}{\partial x_3^2} + C_{23}\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + C_{24}\frac{\partial \phi_2}{\partial x_1} + C_{25}\frac{\partial^2 \phi_2}{\partial x_1 \partial x_3} + C_{26}P^{(s)}\frac{\partial E_3}{\partial x_3} = \rho\frac{\partial^2 u_3}{\partial t^2}, \tag{72}$$

$$C_{31}\frac{\partial u_1}{\partial x_3} + C_{32}\frac{\partial u_3}{\partial x_1} + C_{33}\frac{\partial^2 u_1}{\partial x_1^2} + C_{34}\frac{\partial^2 u_1}{\partial x_3^2} + C_{35}\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + C_{36}\frac{\partial^2 \phi_2}{\partial x_1^2} + C_{37}\frac{\partial^2 \phi_2}{\partial x_3^2} + \\ + C_{38}P^{(s)}\frac{\partial E_3}{\partial x_1} - C_2^{(\theta)}\phi_2 = \rho_2\frac{\partial^2 \phi_2}{\partial t^2}, \tag{73}$$

where the following notations are used:

$$C_{11}=C_{11}^{(\varepsilon)}-\frac{(C_1^{(\delta\varepsilon)})^2}{C^{(\delta)}},\quad C_{12}=C_{55}^{(\varepsilon)}-C_{25}^{(\theta\varepsilon)}+\frac{1}{4}C_2^{(\theta)},\quad C_{13}=C_{13}^{(\varepsilon)}+C_{55}^{(\varepsilon)}-\frac{C_1^{(\delta\varepsilon)}C_3^{(\delta\varepsilon)}}{C^{(\delta)}}-\frac{1}{4}C_2^{(\theta)},$$
$$C_{14}=C_{31}^{(\gamma\varepsilon)}-\frac{C_1^{(\delta\varepsilon)}C_3^{(\delta\gamma)}}{C^{(\delta)}},\quad C_{15}=C_{25}^{(\theta\varepsilon)}-\frac{1}{2}C_2^{(\theta)},\quad C_{16}=\frac{1}{2}C_6^{(\gamma\theta)}-C_{15}^{(\gamma\varepsilon)},\quad C_{17}=\frac{C_1^{(\delta\varepsilon)}}{C^{(\delta)}},$$
$$C_{21}=C_{55}^{(\varepsilon)}+C_{25}^{(\theta\varepsilon)}+\frac{1}{4}C_2^{(\theta)},\quad C_{22}=C_{33}^{(\varepsilon)}-\frac{(C_3^{(\delta\varepsilon)})^2}{C^{(\delta)}},\quad C_{23}=C_{13}^{(\varepsilon)}+C_{55}^{(\varepsilon)}-\frac{C_1^{(\delta\varepsilon)}C_3^{(\delta\varepsilon)}}{C^{(\delta)}}-\frac{1}{4}C_2^{(\theta)},$$
$$C_{24}=C_{25}^{(\theta\varepsilon)}+\frac{1}{2}C_2^{(\theta)},\quad C_{25}=C_{33}^{(\gamma\varepsilon)}-C_{15}^{(\gamma\varepsilon)}-\frac{1}{2}C_6^{(\gamma\theta)}-\frac{C_3^{(\delta\gamma)}C_3^{(\delta\varepsilon)}}{C^{(\delta)}},\quad C_{26}=\frac{C_3^{(\delta\varepsilon)}}{C^{(\delta)}},$$
$$C_{31}=\frac{1}{2}C_2^{(\theta)}-C_{25}^{(\theta\varepsilon)},\quad C_{32}=-\frac{1}{2}C_2^{(\theta)}-C_{25}^{(\theta\varepsilon)},\quad C_{33}=C_{31}^{(\gamma\varepsilon)}-\frac{C_3^{(\delta\gamma)}C_1^{(\delta\varepsilon)}}{C^{(\delta)}},$$
$$C_{34}=\frac{1}{2}C_6^{(\gamma\theta)}-C_{15}^{(\gamma\varepsilon)},\quad C_{35}=C_{33}^{(\gamma\varepsilon)}-C_{15}^{(\gamma\varepsilon)}-\frac{1}{2}C_6^{(\gamma\theta)}-\frac{C_3^{(\delta\gamma)}C_3^{(\delta\varepsilon)}}{C^{(\delta)}}=C_{25},$$
$$C_{36}=C_3^{(\gamma)}-\frac{(C_3^{(\delta\gamma)})^2}{C^{(\delta)}},\quad C_{37}=C_1^{(\gamma)},\quad C_{38}=\frac{C_3^{(\delta\gamma)}}{C^{(\delta)}},\quad \rho_2=\rho J_2.$$

Let us perform similar manipulation for the set of classical equations (1), (2), (6) and (7). After transformations the system looks as follows:

$$C_{11}^{(\varepsilon)}\frac{\partial^2u_1}{\partial x_1^2}+C_{55}^{(\varepsilon)}\frac{\partial^2u_1}{\partial x_3^2}+(C_{13}^{(\varepsilon)}+C_{55}^{(\varepsilon)})\frac{\partial^2u_3}{\partial x_1\partial x_3}-M_{15}\frac{\partial E_1}{\partial x_3}-M_{31}\frac{\partial E_3}{\partial x_1}=\rho\frac{\partial^2u_1}{\partial t^2},\quad (74)$$

$$C_{55}^{(\varepsilon)}\frac{\partial^2u_3}{\partial x_1^2}+C_{33}^{(\varepsilon)}\frac{\partial^2u_3}{\partial x_3^2}+(C_{13}^{(\varepsilon)}+C_{55}^{(\varepsilon)})\frac{\partial^2u_1}{\partial x_1\partial x_3}-M_{15}\frac{\partial E_1}{\partial x_1}-M_{33}\frac{\partial E_3}{\partial x_3}=\rho\frac{\partial^2u_3}{\partial t^2},\quad (75)$$

$$M_{15}\frac{\partial^2u_3}{\partial x_1^2}+M_{33}\frac{\partial^2u_3}{\partial x_3^2}+(M_{15}+M_{31})\frac{\partial^2u_1}{\partial x_1\partial x_3}+\epsilon_1\frac{\partial E_1}{\partial x_1}+\epsilon_3\frac{\partial E_3}{\partial x_3}=0.\quad (76)$$

Now let us consider another two-dimensional case. Let material be as in the previous case, with two orthogonal planes of symmetry. Let thickness be along $\mathbf{e}_3$ direction. Then, for displacements we have equations:

$$\mathbf{u}=u_1(x_1,x_3)\mathbf{e}_1+u_3(x_1,x_3)\mathbf{e}_3,\qquad \boldsymbol{\phi}=\phi_2(x_1,x_3)\mathbf{e}_2.\quad (77)$$

Let the spontaneous polarization be along the $\mathbf{e}_1$ direction and electric field is parallel to the thickness direction:

$$\mathbf{P}^{(s)}=P^{(s)}\mathbf{e}_1,\qquad \mathbf{E}=E_3(x_1,x_3)\mathbf{e}_3.$$

Similarly to (71)–(73), the following system of equation is obtained:

$$C_{11}\frac{\partial^2u_1}{\partial x_1^2}+C_{12}\frac{\partial^2u_1}{\partial x_3^2}+C_{13}\frac{\partial^2u_3}{\partial x_1\partial x_3}+C_{14}\frac{\partial^2\phi_2}{\partial x_1^2}+C_{15}\frac{\partial\phi_2}{\partial x_3}+C_{16}\frac{\partial^2\phi_2}{\partial x_3^2}=0,\quad (78)$$

$$C_{21}\frac{\partial^2u_3}{\partial x_1^2}+C_{22}\frac{\partial^2u_3}{\partial x_3^2}+C_{23}\frac{\partial^2u_1}{\partial x_1\partial x_3}+C_{24}\frac{\partial\phi_2}{\partial x_1}+C_{25}\frac{\partial^2\phi_2}{\partial x_1\partial x_3}=0,\quad (79)$$

$$C_{31}\frac{\partial u_1}{\partial x_3}+C_{32}\frac{\partial u_3}{\partial x_1}+C_{33}\frac{\partial^2 u_1}{\partial x_1^2}+C_{34}\frac{\partial^2 u_1}{\partial x_3^2}+C_{35}\frac{\partial^2 u_3}{\partial x_1\partial x_3}+C_{36}\frac{\partial^2 \phi_2}{\partial x_1^2}+C_{37}\frac{\partial^2 \phi_2}{\partial x_3^2}+ \\ +\left(C_1^{(\delta\varepsilon)}\frac{\partial u_1}{\partial x_1}+C_3^{(\delta\varepsilon)}\frac{\partial u_3}{\partial x_3}+C_3^{(\delta\gamma)}\frac{\partial \phi_2}{\partial x_1}\right)\frac{P^{(s)}}{C^{(\delta)}}E_3-P^{(s)}E_3-C_2^{(\theta)}\phi_2=0. \quad (80)$$

Equations (78)–(80) differs from equations (71)–(73) by the presence of terms, proportional to electric field in equation (80). This situation occurs due to the geometry of the case: spontaneous dipole momentum is perpendicular to electric field direction. The set of classical equations is written as follows:

$$C_{11}^{(\varepsilon)}\frac{\partial^2 u_1}{\partial x_1^2}+C_{55}^{(\varepsilon)}\frac{\partial^2 u_1}{\partial x_3^2}+(C_{13}^{(\varepsilon)}+C_{55}^{(\varepsilon)})\frac{\partial^2 u_3}{\partial x_1\partial x_3}-M_{31}\frac{\partial E_3}{\partial x_1}=\rho\frac{\partial^2 u_1}{\partial t^2}, \quad (81)$$

$$C_{55}^{(\varepsilon)}\frac{\partial^2 u_3}{\partial x_1^2}+C_{33}^{(\varepsilon)}\frac{\partial^2 u_3}{\partial x_3^2}+(C_{13}^{(\varepsilon)}+C_{55}^{(\varepsilon)})\frac{\partial^2 u_1}{\partial x_1\partial x_3}-M_{33}\frac{\partial E_3}{\partial x_3}=\rho\frac{\partial^2 u_3}{\partial t^2}, \quad (82)$$

$$M_{15}\frac{\partial^2 u_3}{\partial x_1^2}+M_{33}\frac{\partial^2 u_3}{\partial x_3^2}+(M_{15}+M_{31})\frac{\partial^2 u_1}{\partial x_1\partial x_3}+\epsilon_3\frac{\partial E_3}{\partial x_3}=0. \quad (83)$$

The comparison of two systems of equations (78)–(80) and (81)–(83) shows significant difference of two theories. In the classical theory electric field comes into equations only via its derivatives, while in the micro-polar theory we find linear terms.

The obtained equations are rather complicated. Meanwhile, it is possible to reduce number of independent variables by using assumption for Cosserat medium ($\boldsymbol{\phi}=0$). From this point of view we are going to consider the following, easy case.

7 The solution for one-dimensional static case.

Let us consider the infinite plate in electric field. Let the material, from which the plate is made of, have two mirror planes of symmetry, crossed by $\mathbf{e}_3$ axis. This symmetry group is named mm2 and it represent such materials as $LiGaO_2$ and Li_2GeO_3. One of the planes of the mirror symmetry is parallel to the plane of the plate. Let spontaneous polarization lay in the same direction and electric field is parallel to the thickness along $\mathbf{n}=\mathbf{e}_2$ axis:

$$\mathbf{P}^{(s)}=P^{(s)}\mathbf{e}_3, \qquad \mathbf{E}=E_2\mathbf{e}_2.$$

The only independent variable is x_2. From equations (54)–(55), after transformations, it follows:

$$\left(C_{22}^{(\varepsilon)}-\frac{(C_2^{(\delta\varepsilon)})^2}{C^{(\delta)}}\right)\frac{\partial^2 u_2}{\partial x_2^2}=0, \quad (84)$$

$$\left(4C_{44}^{(\varepsilon)}-4C_{14}^{(\theta\varepsilon)}+C_1^{(\theta)}\right)\frac{\partial^2 u_3}{\partial x_2^2}=0, \quad (85)$$

$$\left(\frac{1}{2}C_1^{(\theta)}-C_{14}^{(\theta\varepsilon)}\right)\frac{\partial u_3}{\partial x_2}+\left(\frac{C_2^{(\delta\varepsilon)}C_3^{(\delta\gamma)}}{C^{(\delta)}}-C_{32}^{(\gamma\varepsilon)}\right)\frac{\partial^2 u_2}{\partial x_2^2}=P^{(s)}E_2. \quad (86)$$

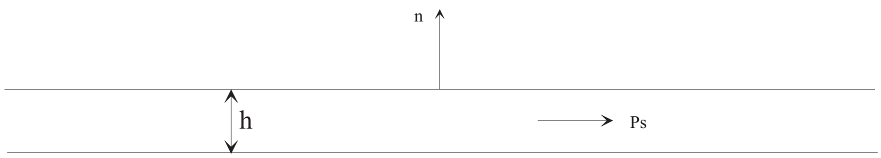

Figure 2: one-dimensional case

Choose the boundary conditions as follows:

$$x_2 = \pm h/2: \qquad \mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n} = 0, \tag{87}$$

and

$$x_2 = 0: \qquad u_2 = 0, \qquad u_3 = 0. \tag{88}$$

After transformations, (87) becomes:

$$x_2 = \pm h/2: \qquad \frac{\partial u_2}{\partial x_2} = 0. \tag{89}$$

Using (88), for u_2 we have:

$$u_2(x_2) = 0. \tag{90}$$

From (86) and (88) the solution for u_3 follows:

$$u_3(x_2) = P^{(s)}E_2x_2 \Big/ \left(\tfrac{1}{2}C_1^{(\theta)} - C_{14}^{(\theta\varepsilon)}\right). \tag{91}$$

7.1 The solution for one-dimensional classical case.

Let us consider equations (6)–(7). We will use here electric potential φ instead of electric field $\mathbf{E} = -\boldsymbol{\nabla}\varphi$. The system of equations in this case can be expressed by:

$$\frac{\partial^2 u_2}{\partial x_2^2} = 0, \qquad C_{44}\frac{\partial^2 u_3}{\partial x_2^2} + M_{24}\frac{\partial^2 \varphi}{\partial x_2^2} = 0, \qquad M_{24}\frac{\partial^2 u_3}{\partial x_2^2} - \epsilon_2\frac{\partial^2 \varphi}{\partial x_2^2} = 0. \tag{92}$$

The boundary conditions:

$$x_2 = \pm h/2: \qquad \mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n} = 0, \qquad \mathbf{n}\cdot\mathbf{D} = \mathbf{n}\cdot\mathbf{D}_0,$$

$$x_2 = 0: \qquad u_2 = 0, \qquad u_3 = 0, \qquad \varphi = 0,$$

where $\mathbf{D}_0 = \epsilon_0\mathbf{E}$, ϵ_0 is the dielectric permittivity of vacuum. The boundary conditions are as follows:

$$\frac{\partial u_2}{\partial x_2} = 0, \qquad C_{44}\frac{\partial u_3}{\partial x_2} + M_{24}\frac{\partial \varphi}{\partial x_2} = 0, \qquad M_{24}\frac{\partial u_3}{\partial x_2} - \epsilon_2\frac{\partial \varphi}{\partial x_2} = E_2.$$

The solution of this system is as follows:

$$u_2 = 0, \qquad u_3 = \frac{M_{24}}{M_{24}^2 + C_{44}\epsilon_2}E_2x_2, \qquad \varphi = -\frac{C_{44}}{M_{24}^2 + C_{44}\epsilon_2}E_2x_2. \tag{93}$$

The solution for displacement $\mathbf{u}$ (90)–(91) and the solution (93) differs by constant multiplier. Both solutions contain material constants, but in the first case that constants are not yet known. Thus, the solutions seems to be equivalent.

8 Conclusion.

The micropolar theory of piezoelectricity has some important advantages compared to the classical one. Considered theory clearly shows the way how electric field influence on matter. There is possibility to consider inhomogeneous mediums by setting $\mathcal{P}^{s}(\mathbf{r})$ field. The micropolar theory allow to consider more general cases then classical one, adding new degrees of freedom. There is possibility to greatly simplify the micropolar theory by neglecting rotational degrees of freedom. Even after that simplification theory remains unsymmetrical and, thus, generally different compared to classic one. Meanwhile, unsymmetrical linear theory may lead to similar material tensor shapes and solutions, obtained by both theories.

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